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ON THE CARTESIAN OVAL (*Second Paper.*)

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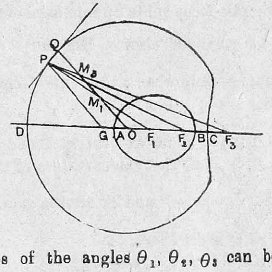
Trigonometric Analysis.

1. Let O be the triple focus and F_1, F_2, F_3 the single foci of a cartesian oval (Γ), in regard to which we shall generally follow the notation in the previous paper (*J. I. M. S.*, Vol. XI, pp. 123-144.)

Let P be a point on the oval of parameter λ and let θ_1, θ_2, θ_3, denote the angles that F_1P, F_2P, F_3P, make with the axis pointed leftwards, that is, with F_1O, F_2O, F_3O. The trigonometric analysis of these angles leads to a number of interesting results.



2. The cosines of the angles θ_1, θ_2, θ_3 can be found from the triangles OF_1P, OF_2P, OF_3P, whose sides are known. For OF_1, OF_2, OF_3 are a, b, c, which we may speak of as the *focal lengths* of the oval. The side $OP = \sqrt{bc + ca + ab + 2\lambda}$ (Section III, *First Paper*). The lengths F_1P, F_2P, F_3P, are ρ_1, ρ_2, ρ_3, when P is on the outer oval, where $\rho_1 = \sqrt{bc} + \lambda/\sqrt{bc}$, &c. If P be on the inner oval, the

lengths are ρ_1, $-\rho_2$, $-\rho_3$ (Section II, *First Paper.*) On account of this difference, our formulæ will come to differ for the two ovals. But we can avoid this by a simple convention, that is, we shall regard $\sqrt{a}$ (the square root of the smallest focal length) to be intrinsically negative for the inner oval. To put it more fully, while $\sqrt{a}$ $\sqrt{b}$, $\sqrt{c}$, are all taken positive for the outer oval, we shall regard $\sqrt{a}$ as negative, while $\sqrt{b}$ and $\sqrt{c}$ are positive, for the inner oval. Further, we shall understand $\sqrt{ab}$ as short for $\sqrt{a}\sqrt{b}$, and $\sqrt{abc}$ as short for $\sqrt{a}\sqrt{b}\sqrt{c}$, and so on. On this understanding it is clear that the lengths F_1P, F_2P, F_3P, taken positively, are ρ_1, ρ_2, ρ_3, for both the ovals ; and we obtain

$$\left.\begin{aligned}\cos\theta_1 &= \left(\frac{\lambda^2}{abc}+a-b-c\right)\div 2\rho_1\\ \cos\theta_2 &= \left(\frac{\lambda^2}{abc}+b-c-a\right)\div 2\rho_2\\ \cos\theta_3 &= \left(\frac{\lambda^2}{abc}+c-a-b\right)\div 2\rho_3.\end{aligned}\right\}\quad\ldots\quad\ldots\quad(1)$$

3. From these equations we find

$$\left.\begin{aligned}\sin\tfrac{1}{2}\theta_1 &= \tfrac{1}{2}\sqrt{\frac{V_1V_4}{\rho_1}}\,;\ \cos\tfrac{1}{2}\theta_1 = \tfrac{1}{2}\sqrt{\frac{V_2V_3}{\rho_1}}\,;\\ \sin\tfrac{1}{2}\theta_2 &= \tfrac{1}{2}\sqrt{\frac{V_2V_4}{\rho_2}}\,;\ \cos\tfrac{1}{2}\theta_2 = \tfrac{1}{2}\sqrt{\frac{V_3V_1}{\rho_2}}\,;\\ \sin\tfrac{1}{2}\theta_3 &= \tfrac{1}{2}\sqrt{\frac{V_3V_4}{\rho_3}}\,;\ \cos\tfrac{1}{2}\theta_3 = \tfrac{1}{2}\sqrt{\frac{V_1V_2}{\rho_3}},\end{aligned}\right\}\quad\ldots\quad(2)$$

where V_1, V_2, V_3, V_4 denote four new linear functions of λ as follows :—

$$\left.\begin{aligned}V_1 &= \frac{\lambda}{\sqrt{abc}}+\sqrt{b}+\sqrt{c}-\sqrt{a}\\ V_2 &= \frac{\lambda}{\sqrt{abc}}+\sqrt{c}+\sqrt{a}-\sqrt{b}\\ V_3 &= \frac{\lambda}{\sqrt{abc}}+\sqrt{a}+\sqrt{b}-\sqrt{c}\\ V_4 &= \sqrt{a}+\sqrt{b}+\sqrt{c}-\frac{\lambda}{\sqrt{abc}}.\end{aligned}\right\}\quad\ldots\quad(3)$$

Hence we get

$$\tan\tfrac{1}{2}\theta_1 = \sqrt{\frac{V_1V_4}{V_2V_3}},\ \tan\tfrac{1}{2}\theta_2 = \sqrt{\frac{V_2V_4}{V_3V_1}},\ \tan\tfrac{1}{2}\theta_3 = \sqrt{\frac{V_3V_4}{V_1V_2}}\quad\ldots\quad(4)$$

We also get

$$\left.\begin{aligned} 4\rho_1 &= V_1 V_4 + V_2 V_3 \\ 4\rho_2 &= V_2 V_4 + V_3 V_1 \\ 4\rho_3 &= V_3 V_4 + V_1 V_2. \end{aligned}\right\} \quad \ldots \quad (5)$$

Let Y denote the perpendicular from P on the axis.

Then $\quad Y = \rho_1 \sin \theta_1 = 2\rho_1 \sin \tfrac{1}{2}\theta_1 \cos \tfrac{1}{2}\theta_1,$

which gives $\quad Y = \tfrac{1}{2}\sqrt{V_1 V_2 V_3 V_4} \quad \ldots \quad \ldots \quad \ldots \quad (6)$

4. It is desirable to note at this stage that the V's (like the ρ's) are all positive, for points on the outer, as well as, the inner oval. For, from equations (2), since the ρ's are all positive, it is seen at once that the V's are all of one sign. And this sign can be seen to be positive by examining that of V_4, say, for each of the ovals.

5. From formulæ (2), we next find that

$$\left.\begin{aligned} \sin \tfrac{1}{2}(\theta_2 - \theta_3) &= \tfrac{1}{2}\sqrt{\frac{V_1 V_4}{\rho_2 \rho_3}}(\sqrt{c} - \sqrt{b}) ; \\ \cos \tfrac{1}{2}(\theta_2 - \theta_3) &= \tfrac{1}{2}\sqrt{\frac{V_2 V_3}{\rho_2 \rho_3}}(\sqrt{c} + \sqrt{b}) ; \\ \text{hence} \quad \tan \tfrac{1}{2}(\theta_2 - \theta_3) &= \sqrt{\frac{V_1 V_4}{V_2 V_3}} \cdot \frac{\sqrt{c} - \sqrt{b}}{\sqrt{c} + \sqrt{b}} ; \end{aligned}\right\} \quad \ldots \quad (7)$$

with similar expressions for the sines, cosines and tangents of $\tfrac{1}{2}(\theta_1 - \theta_3)$ and $\tfrac{1}{2}(\theta_1 - \theta_2)$.

Hence we get

$$\left.\begin{aligned} \frac{\tan \tfrac{1}{2}(\theta_2 - \theta_3)}{\tan \tfrac{1}{2}\theta_1} &= \frac{\sqrt{c} - \sqrt{b}}{\sqrt{c} + \sqrt{b}} \\ \frac{\tan \tfrac{1}{2}(\theta_1 - \theta_3)}{\tan \tfrac{1}{2}\theta_2} &= \frac{\sqrt{c} - \sqrt{a}}{\sqrt{c} + \sqrt{a}} \\ \frac{\tan \tfrac{1}{2}(\theta_1 - \theta_2)}{\tan \tfrac{1}{2}\theta_3} &= \frac{\sqrt{b} - \sqrt{a}}{\sqrt{b} + \sqrt{a}}. \end{aligned}\right\} \quad \ldots \quad (8)$$

Thus the expressions on the left continue constant as P moves on the oval.

6. We next proceed to prove the important relations

$$\frac{Y}{\sqrt{\rho_1 \rho_2 \rho_3}} = \frac{\sin \tfrac{1}{2}(\theta_1 + \theta_2 + \theta_3)}{\dfrac{\lambda}{\sqrt{abc}}} = \frac{\sin \tfrac{1}{2}(\theta_2 + \theta_3 - \theta_1)}{\sqrt{a}}$$

$$= \frac{\sin \tfrac{1}{2}(\theta_3 + \theta_1 - \theta_2)}{\sqrt{b}} = \frac{\sin \tfrac{1}{2}(\theta_1 + \theta_2 - \theta_3)}{\sqrt{c}} \quad \ldots \quad (9)$$

Let $A = \sin\frac{1}{2}\theta_1 \,.\, \cos\frac{1}{2}\theta_2 \,.\, \cos\frac{1}{2}\theta_3$
$B = \cos\frac{1}{2}\theta_1 \,.\, \sin\frac{1}{2}\theta_2 \,.\, \cos\frac{1}{2}\theta_3$
$C = \cos\frac{1}{2}\theta_1 \,.\, \cos\frac{1}{2}\theta_2 \,.\, \sin\frac{1}{2}\theta_3$
$D = \sin\frac{1}{2}\theta_1 \,.\, \sin\frac{1}{2}\theta_2 \,.\, \sin\frac{1}{2}\theta_3$.

Then we have

$$\sin\tfrac{1}{2}(\theta_1 + \theta_2 + \theta_3) = A + B + C - D,$$
$$\sin\tfrac{1}{2}(\theta_2 + \theta_3 - \theta_1) = -A + B + C + D,$$
$$\sin\tfrac{1}{2}(\theta_3 + \theta_1 - \theta_2) = A - B + C + D,$$
$$\sin\tfrac{1}{2}(\theta_1 + \theta_2 - \theta_3) = A + B - C + D,$$

Calculating their values by (2), we get

$$A, B, C, D = \tfrac{1}{8}\sqrt{\frac{V_1V_2V_3V_4}{\rho^1\rho_2\rho_3}}\,(V_1, V_2, V_3, V_4), \text{ respectively.}$$

Therefore we have

$$\sin\tfrac{1}{2}(\theta_1 + \theta_2 + \theta_3) = \tfrac{1}{8}\sqrt{\frac{V_1\,V_2\,V_3\,V_4}{\rho_1\,\rho_2\,\rho_3}}\,(V_1+V_2+V_3-V_4)$$

$$\sin\tfrac{1}{2}(\theta_2 + \theta_3 - \theta_1) = \tfrac{1}{8}\sqrt{\frac{V_1\,V_2\,V_3\,V_4}{\rho_1\,\rho_2\,\rho_3}}\,(V_2+V_3+V_4-V_1)$$

&c.

But from equations (3), we get

$$\frac{\lambda}{\sqrt{abc}} = \tfrac{1}{4}(V_1 + V_2 + V_3 - V_4)$$
$$\sqrt{a} = \tfrac{1}{4}(V_2 + V_3 + V_4 - V_1) \text{ \&c.}$$

Also $$Y = \tfrac{1}{2}\sqrt{V_1V_2V_3V_4}.$$

Hence we get

$$\sin\tfrac{1}{2}(\theta_1 + \theta_2 + \theta_3) = \frac{Y}{\sqrt{\rho_1\rho_2\rho_3}}\,.\,\frac{\lambda}{\sqrt{abc}}$$

$$\sin\tfrac{1}{2}(\theta_2 + \theta_3 - \theta_1) = \frac{Y}{\sqrt{\rho_1\,\rho_2\,\rho_3}}\,.\,\sqrt{a}. \text{ \&c.}$$

which proves the relations stated.

7. The first application we shall make of these equations will be as follows. From O let us draw a line making with the axis pointed leftwards an angle equal to $\frac{1}{2}(\theta_1 + \theta_2 + \theta_3)$ and cutting F_1P, F_2P, F_3P at M_1, M_2, M_3. The line will evidently cut F_1P, F_2P, F_3P at

the angles $\frac{1}{2}(\theta_2 + \theta_3 - \theta_1)$, $\frac{1}{2}(\theta_3+\theta_1 - \theta_2)$ $\frac{1}{2}(\theta_1 + \theta_2 - \theta_3)$. Now let us see into what parts the radius vector F_1P, for example, is cut at M_1.

From the triangle OF_1M_1 we have

$$\frac{M_1F_1}{F_1O} = \frac{\sin\frac{1}{2}(\theta_1 + \theta_2 + \theta_3)}{\sin\frac{1}{2}(\theta_2 + \theta_3 - \theta_1)}$$

$$= \frac{\dfrac{\lambda}{\sqrt{abc}}}{\sqrt{a}} = \frac{\dfrac{\lambda}{\sqrt{bc}}}{a}.$$

But $F_1O = a$. Hence $M_1F_1 = \lambda/\sqrt{bc}$; and thence we get
$$P_1M_1 = \sqrt{bc}.$$

The first property of the triple focal perpendicular on the tangent enables us at once to identify the line. It is *itself* the triple focal perpendicular on the tangent at P.

We have thus proved the *fourth* property of the triple focal perpendicular on the tangent, namely, it makes with the axis pointed leftwards an angle equal to one-half of the sum of the angles that the focal vectors F_1P, F_2P, F_3P make with the axis pointed leftwards.

We further see that it cuts F_1P, F_2P, F_3P at the angles $\frac{1}{2}(\theta_2 + \theta_3 - \theta_1)$, $\frac{1}{2}(\theta_3 + \theta_1 - \theta_2)$, $\frac{1}{2}(\theta_1 + \theta_2 - \theta_3)$.

8. Draw the normal at P cutting the axis in G. We see that the normal GP makes with the axis pointed leftwards an angle equal to $\frac{1}{2}(\theta_1 + \theta_2 + \theta_3)$.

Hence it follows that confocal cartesians intersect orthogonally.

Further we see that if a point P describe a cartesian oval the angular velocity of the normal at P is one-half of the sum of the angular velocities of the three focal vectors of P.*

9. Let Q be the foot of the perpendicular from O on the tangent at P; and let $PQ = t$. From the triangles PM_1Q, PM_2Q, PM_3Q, we get

$$\frac{t}{\sqrt{abc}} = \frac{\sin\frac{1}{2}(\theta_2 + \theta_3 - \theta_1)}{\sqrt{a}} = \&c. \ldots \qquad \ldots (10)$$

That is, we have the term $\dfrac{t}{\sqrt{abc}}$ added to the list of equal quantities in (9).

* This property holds for bi-circular quartics, as it holds for the bifocal conics and the parabola, with reference to the number of focal vectors possessed in each case, and leads at once to formulæ for the radius of curvature.

In particular, we have

$$\frac{t}{\sqrt{abc}} = \frac{Y}{\sqrt{\rho_1\rho_2\rho_3}} \quad \dots \quad \dots \quad \dots \quad (11)$$

We shall presently meet with many more terms equal to these in value and, for convenience sake, we shall use the letter M to denote any one of them. Using the value of Y, we have

$$M = \tfrac{1}{2}\sqrt{\frac{V_1 V_2 V_3 V_4}{\rho_1\rho_2\rho_3}}. \quad \dots \quad (11\text{-}a)$$

For the truth of the results (10) and (11) for the inner oval, we should regard t as intrinsically negative, like $\sqrt{a}$, for that oval.

10. Let us consider our results in connection with the confocal cartesian oval (Γ') passing through P. Its triple focus O′ will be a point in OF_3 produced. Let us denote its focal lengths F_1O', F_2O', F_3O' by a', b', c'. The smallest of them is c'. Let the parameter of P in regard to Γ' be λ'; and let the distance OO′ between the triple foci be denoted by δ, so that we have $a + a' = b + b' = c + c' = \delta$. The θ's of the point P in regard to Γ' are the supplements of the θ's we are dealing with. Hence, noting that Y, ρ_1, ρ_2, ρ_3 mean the same geometrical quantities in connection with P for each of the confocals, we get from (9)

$$M = \frac{Y}{\sqrt{\rho_1\rho_3\rho_2}} = \frac{-\cos\frac{1}{2}(\theta_1+\theta_2+\theta_3)}{\dfrac{\lambda'}{\sqrt{a'b'c'}}} = \frac{\cos\frac{1}{2}(\theta_2+\theta_3-\theta_1)}{\sqrt{a'}}$$

$$= \frac{\cos\frac{1}{2}(\theta_3+\theta_1-\theta_2)}{\sqrt{b'}} = \frac{\cos\frac{1}{2}(\theta_1+\theta_2-\theta_3)}{\sqrt{c'}}, \quad \dots \quad (12)$$

which adds four more to our list of equivalent terms.

11. Collating now the results

$$M = \frac{\sin\frac{1}{2}(\theta_2+\theta_3-\theta_1)}{\sqrt{a}} = \frac{\cos\frac{1}{2}(\theta_2+\theta_3-\theta_1)}{\sqrt{a'}}$$

we get, by ratios, $M = \dfrac{1}{\sqrt{a+a'}}$; that is, $M = \dfrac{1}{\sqrt{\delta}}$ ... (13)

which makes an important addition to the set.

Taking this result with $M = Y/\sqrt{\rho_1\rho_2\rho_3}$ we see that the distance between the triple foci of the two confocals is given by

$$\delta = \frac{\rho_1\rho_2\rho_3}{Y^2} \quad \dots \quad \dots \quad \dots \quad (14)$$

12. Taking (13) with $M = \frac{t}{\sqrt{abc}}$, we get

$$t = \sqrt{\frac{abc}{\delta}} \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

Observing that t is equal to the perpendicular from O on the normal at P, as (15) holds for all the eight points of intersection of the two confocals and the right-hand side is the same for all such points, it follows that t is the same for all such points and we have the theorem that the normals to the given cartesian Γ at its eight points of intersection with any confocal Γ' all touch a circle X whose centre is the triple focus O.

The radius of this circle is t which, from the value above obtained is

$$\sqrt{\frac{OF_1 . OF_2 . OF_3}{OO'}} .$$

Hence [from (43), *First Paper*], we infer that the circle X has double contact with the confocal Γ'.

Make now Γ and Γ' change roles. We infer that the tangents to Γ at its eight points of intersection with any confocal cartesian Γ' all touch a circle X' whose centre is the triple focus of Γ' and which has double contact with the given cartesian Γ.*

13. We next proceed to a noteworthy geometrical theorem in regard to the centre of curvature of the oval at P.

Let PG be the normal and S the centre of curvature at P. Then [by (58), *First Paper*] we have

$$G P : S P = (\lambda + t)^2 : \lambda$$

$\therefore$ $$G S : S P = t^2 : \lambda.$$

Now $$G O = abc/\lambda \text{ [by (28), \textit{First Paper}]}$$

and $$OO' = abc/t^2, \text{ by (15).}$$

$\therefore$ $$G O : OO' = t^2 : \lambda.$$

Hence we get $$G O : O O' = G S : S P,$$

whence O S is paralled to P O'. That is, we have the theorem :

In any cartesian oval the line joining the centre of curvature at any point P to the triple focus is parallel to the line joining P to the triple focus of the confocal passing through P.

13·1. In close connection with this we have another theorem, as follows.

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* See Prof. Wilkinson's Paper, *J. I. M. S.* Vol. XI, p. 172.

If Q, Q_1, are the projections of O, O' on the tangent at P, then $PQ.QQ_1 = \lambda$, the parameter of P.

Proof. Since GP, OQ and $O'Q_1$ are parallel, we have

$$PQ : QQ_1 = GO : OO'.$$

Hence $$PQ : QQ_1 = t^2 : \lambda.$$

But $$PQ = t. \quad \therefore \; QQ_1 = \lambda/t.$$

$$\therefore \quad PQ.QQ_1 = \lambda.$$

Remembering that $\lambda = PM_1 . M_1F_1$, &c, this fixes Q_1. Thence we are enabled to fix O' the triple focus of the confocal through P; and thence S the centre of curvature at P, by the foregoing theorem.

14. We next proceed to prove the following relations from which we can derive a good account of Genochhi's Theorem. The relations are

$$\left.\begin{aligned} \sqrt{b}.\cos\theta_2 + \sqrt{c}.\cos\theta_3 + \sqrt{a}\cos(\theta_2 - \theta_3) &= \frac{\lambda}{\sqrt{abc}} \\ \sqrt{c}.\cos\theta_3 + \sqrt{a}.\cos\theta_1 + \sqrt{b}\cos(\theta_1 - \theta_3) &= \frac{\lambda}{\sqrt{abc}} \\ \sqrt{a}.\cos\theta_1 + \sqrt{b}.\cos\theta_2 + \sqrt{c}.\cos(\theta_1 - \theta_2) &= \frac{\lambda}{\sqrt{abc}} \end{aligned}\right\} \quad (16)$$

Proof. These will be true provided they are true when for $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ and $\frac{\lambda}{\sqrt{abc}}$ we substitute $\sin\frac{1}{2}(\theta_2 + \theta_3 - \theta_1)$, $\sin\frac{1}{2}(\theta_3 + \theta_1 - \theta_2)$ $\sin\frac{1}{2}(\theta_1 + \theta_2 - \theta_3)$ and $\sin\frac{1}{2}(\theta_1 + \theta_2 + \theta_3)$ which are proportional to them. On substitution, the results are seen to become pure trigonometric identities.

15. To deduce Genochhi's Theorem, we require some development of the system of quantities equivalent to M.

We saw that

$$M = \frac{t}{\sqrt{abc}} = \frac{\sin\frac{1}{2}(\theta_2 + \theta_3 - \theta_1)}{\sqrt{a}}$$

$$= \frac{\sin\frac{1}{2}(\theta_3 + \theta_1 - \theta_2)}{\sqrt{b}} = \frac{\sin\frac{1}{2}(\theta_1 + \theta_2 - \theta_3)}{\sqrt{c}}.$$

Hence, by ratios, and use of the identical relations

$$\sin^2(x + y) = \sin^2 x + \sin^2 y + 2\sin x \sin y \cos(x + y)$$

$$\sin^2(x - y) = \sin^2 x + \sin^2 y - 2\sin x \sin y \cos(x - y)$$

we get

$$
\left.\begin{aligned}
M = \frac{t}{\sqrt{abc}} &= \frac{\sin\theta_1}{(b + c + 2\sqrt{bc}\cos\theta_1)^{\frac{1}{2}}} \\
&= \frac{\sin\theta_2}{(c + a + 2\sqrt{ca}\cos\theta_2)^{\frac{1}{2}}} \\
&= \frac{\sin\theta_3}{(a + b + 2\sqrt{ab}\cos\theta_3)^{\frac{1}{2}}} \\
&= \frac{\sin(\theta_2 - \theta_3)}{(b + c - 2\sqrt{bc}\cos\overline{\theta_2 - \theta_3})^{\frac{1}{2}}} \\
&= \frac{\sin(\theta_1 - \theta_3)}{(c + a - 2\sqrt{ca}\cos\overline{\theta_1 - \theta_3})^{\frac{1}{2}}} \\
&= \frac{\sin(\theta_1 - \theta_2)}{(a + b - 2\sqrt{ab}\cos\overline{\theta_1 - \theta_2})^{\frac{1}{2}}}
\end{aligned}\right\} \quad \ldots (17)
$$

which may be considered to give so many expressions for t in terms of the θ's severally, and of their differences severally.

16. Now let P be a point of the outer oval and let the arc DP be equal to s. Then $\theta_1, \theta_2, \theta_3$ are the angles subtended by this arc at the foci.

When s varies, we have

$$\frac{d\lambda}{ds} = - t.$$

This is the little equation we got in (25), *First Paper*, the negative sign coming in here as the arc is measured from D, and not C.

From the relations (16), taking differentials, and noting $d\lambda = \frac{d\lambda}{ds} \cdot ds$,

we get

$$
\left.\begin{aligned}
&\sqrt{b}\,.\sin\theta_2\,.\,d\theta_2 + \sqrt{c}\,.\sin\theta_3\,d\theta_3 + \\
&\qquad \sqrt{a}\sin(\theta_2 - \theta_3)\,.\,d(\theta_2 - \theta_3) = \frac{t}{\sqrt{abc}}\,.\,ds. \\
&\sqrt{c}\,.\sin\theta_3\,d\theta_3 + \sqrt{a}\sin\theta_1\,.\,d\theta_1 + \\
&\qquad \sqrt{b}\sin(\theta_1 - \theta_3)\,.\,d(\theta_1 - \theta_3) = \frac{t}{\sqrt{abc}}\,.\,ds. \\
&\sqrt{a}\,.\sin\theta_1\,d\theta_1 + \sqrt{b}\sin\theta_2\,d\theta_2 + \\
&\qquad \sqrt{c}\sin(\theta_1 - \theta_2)\,.\,d(\theta_1 - \theta_2) = \frac{t}{\sqrt{abc}}\,.\,ds.
\end{aligned}\right\} \quad \ldots (18)
$$

Substitute now for $\sin\theta_1$, $\sin\theta_2$, $\sin\theta_3$, $\sin(\theta_2-\theta_3)$, $\sin(\theta_1-\theta_3)$, $\sin(\theta_1-\theta_2)$ and t, in the above, the quantities proportional to them

forming the denominators in the relations (17). We get

$$\left.\begin{aligned} ds &= \sqrt{b}\,(c + a + 2\sqrt{ca}\cos\theta_2)^{\frac{1}{2}}\,d\theta_2 \\ &\quad + \sqrt{c}\,(a + b + 2\sqrt{ab}\cos\theta_3)^{\frac{1}{2}}\,d\theta_3 \\ &\quad + \sqrt{a}\,(b + c - 2\sqrt{bc}\cos\overline{\theta_2 - \theta_3})^{\frac{1}{2}}\,d(\theta_2 - \theta_3). \\ ds &= \sqrt{c}\,(a + b + 2\sqrt{ab}\cos\theta_3)^{\frac{1}{2}}\,d\theta_3 \\ &\quad + \sqrt{a}\,(b + c + 2\sqrt{bc}\cos\theta_1)^{\frac{1}{2}}\,d\theta_1 \\ &\quad + \sqrt{b}\,(c + a - 2\sqrt{ca}\cos\overline{\theta_1 - \theta_3})^{\frac{1}{2}}\,d(\theta_1 - \theta_3). \\ ds &= \sqrt{a}\,(b + c + 2\sqrt{bc}\,.\cos\theta_1)^{\frac{1}{2}}\,d\theta_1 \\ &\quad + \sqrt{b}\,(c + a + 2\sqrt{ca}\cos\theta_2)^{\frac{1}{2}}\,d\theta_2 \\ &\quad + \sqrt{c}\,(a + b - 2\sqrt{ab}\cos\overline{\theta_1 - \theta_2})^{\frac{1}{2}}\,d(\theta_1 - \theta_2) \end{aligned}\right\} \quad \ldots (19)$$

Hence, by integration, we have

$$\begin{aligned} s = &\int_0^{\theta_2} \sqrt{b}\,(c + a + 2\sqrt{ca}\cos\theta)^{\frac{1}{2}}\,d\theta \\ + &\int_0^{\theta_3} \sqrt{c}\,(a + b + 2\sqrt{ab}\cos\theta)^{\frac{1}{2}}\,d\theta \\ + &\int_0^{\theta_2 - \theta_3} \sqrt{a}\,(b + c - 2\sqrt{bc}\cos\theta)^{\frac{1}{2}}\,d\theta \end{aligned} \quad \ldots (20)$$

and two similar results, proving Genochhi's theorem that the arc of the Cartesian can be expressed in terms of three elliptic arcs; and we see that there are three ways of doing so.

17. By a slightly different treatment we can obtain an expression for s in which the integrals involved are symmetric in $\theta_1, \theta_2, \theta_3$, but which involves an extra term. The first relation in (16) is

$$\sqrt{b}\cos\theta_2 + \sqrt{c}\cos\theta_3 + \sqrt{a}\cos(\theta_2 - \theta_3) = \frac{\lambda}{\sqrt{abc}}.$$

This may be written

$$\begin{aligned} &\sqrt{a}\cos\theta_1 + \sqrt{b}\cos\theta_2 + \sqrt{c}\cos\theta_3 \\ &\qquad + \sqrt{a}\,\{\cos(\theta_2 - \theta_3) - \cos\theta_1\} = \frac{\lambda}{\sqrt{abc}} \end{aligned}$$

Now

$$\begin{aligned} &\sqrt{a}\,\{\cos(\theta_2 - \theta_3) - \cos\theta_1\} \\ &\qquad = 2\sqrt{a}\,\sin\tfrac{1}{2}(\theta_2 + \theta_1 - \theta_3)\sin\tfrac{1}{2}(\theta_1 + \theta_3 - \theta_2) \\ &\qquad = 2\sqrt{a}\,.\,\frac{t}{\sqrt{ab}}\,.\,\frac{t}{\sqrt{ac}} = \frac{2t^2}{\sqrt{abc}}. \end{aligned}$$

Hence we have

$$\sqrt{a}\cos\theta_1 + \sqrt{b}\cos\theta_2 + \sqrt{c}\cos\theta_3 + 2.\frac{t^2}{\sqrt{abc}} = \frac{\lambda}{\sqrt{abc}}$$

Taking differentials, and remembering $\frac{d\lambda}{ds} = -t$, we get

$$\sqrt{a}\,.\sin\theta_1\,.d\theta_1 + \sqrt{b}\,.\sin\theta_2\,.d\theta_2 + \sqrt{c}\,.\sin\theta_3\,.d\theta_3$$

$$-4\frac{t}{\sqrt{abc}}\,.dt = \frac{t}{\sqrt{abc}}\,.ds\,.$$

Substituting for $\sin\theta_1$, $\sin\theta_2$, $\sin\theta_3$ and t the quantities proportional to them from (17) as before, we get

$$ds = \sqrt{a}\,(b+c+2\sqrt{bc}\cos\theta_1)^{\frac{1}{2}}\,d\theta_1$$

$$+\sqrt{b}\,(c+a+2\sqrt{ca}\cos\theta_2)^{\frac{1}{2}}\,d\theta_2$$

$$+\sqrt{c}\,(a+b+2\sqrt{ab}\cos\theta_3)^{\frac{1}{2}}\,d\theta_3 - 4\,dt \quad \dots (21)$$

which gives

$$s = \int_0^{\theta_1}\sqrt{a}\,(b+c+2\sqrt{b}\cos\theta)^{\frac{1}{2}}\,d\theta$$

$$+\int_0^{\theta_2}\sqrt{b}\,(c+a+2\sqrt{ca}\cos\theta)^{\frac{1}{2}}\,d\theta$$

$$+\int_0^{\theta_3}\sqrt{c}\,(a+b+2\sqrt{ab}\cos\theta)^{\frac{1}{2}}\,d\theta - 4t \dots (22)$$

where $t = PQ$ and we know its value, as already observed, in terms of θ_1, θ_2, or θ_3.

18. P being on the outer oval still, let the arc $CP = s'''$ and let us seek to express it in terms of the angles ϕ_1, ϕ_2, ϕ_3 which it subtends at the foci. Since $(s + s''')$ is constant, we have

$$ds = -ds''' \quad \dots \quad \dots \quad \dots (i)$$

Also we have

$$\theta_1 = \pi - \phi_1,\ \theta_2 = \pi - \phi_2,\ \theta_3 = \phi_3\,; \quad \dots (ii)$$

$$d\theta_1 = -d\phi_1,\ d\theta_2 = -d\phi_2,\ d\theta_3 = d\phi_3\,. \quad \dots (iii)$$

By means of these relations we can transform the four expressions for ds contained in (19) and (21) and obtain the corresponding expression s in for ds''' terms of $d\phi_1$, $d\phi_2$, $d\phi^3$, or of these and dt.

19. Let us next consider P to be on the inner oval. Let the arc A P = s'. Then $\theta_1, \theta_2, \theta_3$ will be the angles subtended by this arc at the foci. It can be seen that the investigation for s' is exactly the same as for s. All the steps in the latter as well as the final results hold good ; only, we should consider $\sqrt{a}$ and t to be intrinsically negative for the inner oval. Consequently explicit expressions for ds' are obtained from those for ds in (19) by changing $\sqrt{a}$ into $-\sqrt{a}$; or from that in (21) by changing both $\sqrt{a}$ and t into their negatives.

Finally, we observe that if the arc B P in this case be s'', and it subtend angles ϕ_1, ϕ_2, ϕ_3 at the foci, then expressions for ds'' could be derived from those for ds' in the manner of § 18.

ON THE EXPANSION OF CERTAIN FUNCTIONS*
(WITH PROPERTIES OF ASSOCIATED CO-EFFICIENTS.)

BY C. KRISHNAMACHARI.

Introduction.—Various series involving inverse powers of natural numbers have been summed up by mathematicians. We have the well-known summations in terms of Bernoulli's and Euler's numbers. Mr. S. R. Ranganathan in his paper, "Bernoulli's Polynomial and Fourier Series," (*vide* J. I. M. S., Vol. XI, No. 2), sums a number of interesting series. *The object of this paper is to express the sums of various series* (including those summed by Mr. Ranganathan) *in terms of coefficients of various trigonometric expansions*, which possess interesting properties similar to Bernoulli's and Euler's numbers. Since writing this paper, I find that Dr. Glaisher has anticipated me in the 'Quarterly Journal of Mathematics,' Vol. XXIX, a volume which I have not been able to procure even now from any Indian Library. On page 187 of the Quarterly Journal, Vol. XLV, Dr. Glaisher makes a reference to an article in Vol. XXIX entitled "On the Bernoullian Function." From the title I conclude that Dr. Glaisher's method of proof is different from mine, and that the A_{2n} in § 1 of my paper is Dr. Glaisher's P_n; and my A_{2n-1} is his Q_n. Dr. Glaisher further points out that

$$\frac{2e^{x}}{e^{4x}+1} = 1 - A_1 x - A_2 \cdot \frac{x^2}{2!} + A_3 \cdot \frac{x^3}{3!} + A_4 \cdot \frac{x^4}{4!} - \dots\dots$$

§ 1. Let

$$\frac{1}{\cos x - \sin x} = 1 + A_1 x + A_2 \frac{x^2}{2!} + \dots\dots + A_n \frac{x^n}{n!} + \dots\dots,$$

so that, cross-multiplying and equating to zero the coefficients of the various powers of x, we obtain,

$$\left.\begin{aligned} A_{2n} &= \binom{2n}{1} A_{2n-1} + \binom{2n}{2} A_{2n-2} - \binom{2n}{3} A_{2n-3} \\ &\qquad - \dots\dots + (-1)^{n-1} \binom{2n}{2n-1} A_1 + (-1)^{n-1} \\ A_{2n+1} &= \binom{2n+1}{1} A_{2n} + \binom{2n+1}{2} A_{2n-1} - \binom{2n+1}{3} A_{2n-2} \\ &\qquad - \dots\dots + (-1)^{n-1} \binom{2n+1}{2n} A_1 + (-1)^{n}, \end{aligned}\right\} \quad \dots (1)$$

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* Presented to the Third Conference of the Indian Mathematical Society, March, 1921.

where $\binom{n}{r}$ denotes the number of combinations of n things r at a time.

In particular

$$\left.\begin{array}{llll} A_1 = 1, & A_2 = 3, & A_3 = 11, & A_4 = 57, \\ A_5 = 361, & A_6 = 2763, & A_7 = 24611, & A_8 = 250737. \end{array}\right\}^{*} \quad \ldots (2)$$

Now $$\frac{1}{\cos x - \sin x} = \frac{1}{\sqrt{2}} \cdot \sec\left(\frac{\pi}{4} + x\right)$$

$$= \frac{1}{\sqrt{2}}\left\{\frac{2}{\pi - 2\left(\frac{\pi}{4} + x\right)} + 2\sum_{1}^{\infty}(-1)^{n-1}\left[\frac{1}{2n\pi - \pi + 2\left(\frac{\pi}{4} + x\right)} - \frac{1}{2n\pi + \pi - 2\left(\frac{\pi}{4} + x\right)}\right]\right\}$$

$$= \frac{1}{\sqrt{2}}\left\{\frac{4}{\pi} \cdot \frac{1}{1 - \frac{4x}{\pi}} + \sum_{1}^{\infty}(-1)^{n-1}\left[\frac{4}{(4n-1)\pi} \cdot \frac{1}{1 + \frac{4x}{(4n-1)\pi}} - \frac{4}{(4n+1)\pi} \cdot \frac{1}{1 - \frac{4x}{(4n+1)\pi}}\right]\right\}$$

$$\left.\begin{array}{l} = \frac{1}{\sqrt{2}}\left\{\frac{4}{\pi}\left(1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \ldots\ldots\right)\right. \\ \quad + \frac{4^2}{\pi^2} \cdot x\left(1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \ldots\ldots\right) \\ \quad + \frac{4^3}{\pi^3} \cdot x^2\left(1 + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \frac{1}{11^3} - \ldots\ldots\right) + \\ \quad + \frac{4^{2n-1}}{\pi^{2n-1}} \cdot x^{2n-2}\left(1 + \frac{1}{3^{2n-1}} - \frac{1}{5^{2n-1}} - \frac{1}{7^{2n-1}} + \ldots\ldots\right) \\ \quad \left. + \frac{4^{2n}}{\pi^{2n}} \cdot x^{2n-1}\left(1 - \frac{1}{3^{2n}} - \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{9^{2n}} - \ldots\ldots\right) + \ldots\right\} \end{array}\right\} \ldots (3)$$

Hence by equating the co-efficients of the coresponding powers of x on both sides, we obtain,

$$\left.\begin{array}{l} \frac{A_{2n-1}}{2n-1!} \cdot \left(\frac{\pi}{4}\right)^{2n} \cdot \sqrt{2} = 1 - \frac{1}{3^{2n}} - \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{9^{2n}} - \ldots \\ \frac{A_{2n}}{2n!} \cdot \left(\frac{\pi}{4}\right)^{2n+1} \cdot \sqrt{2} = 1 + \frac{1}{3^{2n+1}} - \frac{1}{5^{2n+1}} - \frac{1}{7^{2n+1}} + \ldots \end{array}\right\}^{\dagger} \quad \ldots (4)$$

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* Dr. Glaisher has tabulated the values of forty of these numbers. *Quarterly Journal*, Vol. XLV. See Introduction.

† These formulæ are arrived at by Dr. Glaisher presumably by a different method. See Introduction.

§ 2. The following generalisation of this method is fruitful of various results in summation of series :

$$\frac{1}{\cos x-\tan\alpha\,.\sin x}=\cos\alpha\,.\,\sec\,(x+\alpha)$$

$$=\cos\alpha\;\left\{\frac{2}{\pi-2\,(x+\alpha)}+2\sum_{1}^{\infty}(-1)^{n-1}\left[\frac{1}{2n\pi-(\pi-2\overline{x+\alpha})}-\frac{1}{2n\pi+\pi-2(x+\alpha)}\right]\right\}$$

$$=2\cos\alpha\;\left\{\frac{1}{\pi-2\alpha}\cdot\frac{1}{1-\dfrac{2x}{\pi-2\alpha}}+\sum_{1}^{\infty}(-1)^{n-1}\right.$$

$$\left.\left[\frac{1}{(2n-1)\,\pi+2\alpha}\cdot\frac{1}{1+\dfrac{2x}{(2n-1)\,\pi+2\alpha}}-\frac{1}{(2n+1)\,\pi-2\alpha}\cdot\frac{1}{1-\dfrac{2x}{(2n+1)\,\pi-2\alpha}}\right]\right\}$$

$$=2\cos\alpha\;\left\{\left(\frac{1}{\pi-2\alpha}+\frac{1}{\pi+2\alpha}-\frac{1}{3\pi-2\alpha}-\frac{1}{3\pi+2\alpha}+\frac{1}{5\pi-2\alpha}+\frac{1}{5\pi+2\alpha}-\dots\right)\right.$$

$$+2x\left(\frac{1}{(\pi-2\alpha)^2}-\frac{1}{(\pi+2\alpha)^2}-\frac{1}{(3\pi-2\alpha)^2}+\frac{1}{(3\pi+2\alpha)^2}+\frac{1}{(5\pi-2\alpha)^2}-\dots\right)$$

$$+2^2x^2\left[\frac{1}{(\pi-2\alpha)^3}+\frac{1}{(\pi+2\alpha)^3}-\frac{1}{(3\pi-2\alpha)^3}-\frac{1}{(3\pi+2\alpha)^3}+\dots\right]$$

$$+\dots\dots\dots\dots\dots\dots\dots\dots$$

$$+2^{2n-1}x^{2n-1}\left[\frac{1}{(\pi-2\alpha)^{2n}}-\frac{1}{(\pi+2\alpha)^{2n}}-\frac{1}{(3\pi-2\alpha)^{2n}}+\frac{1}{(3\pi+2\alpha)^{2n}}+\frac{1}{(5\pi-2\alpha)^{2n}}-\dots\right]$$

$$+2^{2n}x^{2n}\left[\frac{1}{(\pi-2\alpha)^{2n+1}}+\frac{1}{(\pi+2\alpha)^{2n+1}}-\frac{1}{(3\pi-2\alpha)^{2n+1}}-\frac{1}{(3\pi+2\alpha)^{2n+1}}+\dots\right]$$

$$\left.+\;\dots\quad\dots\quad\dots\quad\dots\quad\dots\right\}\qquad(5)$$

after expanding the partial fractions and re-arranging in powers of x, We have taken the absolutely convergent series for sec θ.

Hence by writing the expansion in the form

$$1 + \sum A_n (\tan \alpha) \frac{x^n}{n!},$$

and equating the co-efficients of the powers of x, we have ... (6)

$$\frac{A_{2n-1} (\tan \alpha)}{(2n-1)!} = 2^{2} \cos \alpha \left[\frac{1}{(\pi - 2\alpha)^{2n}} - \frac{1}{(\pi + 2\alpha)^{2n}} - \frac{1}{(3\pi - 2\alpha)^{2n}} + \frac{1}{(3\pi + 2\alpha)^{2n}} + \dots \right] \quad \dots (7)$$

$$\frac{A_{2n} (\tan \alpha)}{2n!} = 2^{2n+1} \cos \alpha \left[\frac{1}{(\pi - 2\alpha)^{2n+1}} + \frac{1}{(\pi + 2\alpha)^{2n+1}} - \frac{1}{(3\pi - 2\alpha)^{2n+1}} - \frac{1}{(3\pi + 2\alpha)^{2n+1}} + \dots \right] \quad \dots (8)$$

In a paper on "A Table of Values of thirty Eulerian Numbers based on a New Method" by Mr. M. Bheemasena Rao and myself, it is proved that, if we write, tan $\alpha = a$, then we have

$$\left.\begin{aligned} A_n (\tan \alpha) &= F(a) \\ &= n!\, a^n + (n-2)! \sum (n-1)^2 a^{n-2} \\ &+ (n-4)! \sum (n-3)^2 \sum (n-2)^2 a^{n-4} + \dots \\ &+ (n-2r)! \sum (n-2r+1)^2 \sum (n-2r+2)^2 \dots \\ &\qquad \sum (n-r)^2 a^{n+2r} + \dots \end{aligned}\right\} \quad (9)$$

where the '$\sum$' notation is used in a particular sense.

The above equations are fundamental, and a number of interesting summations can be obtained by giving particular values to α. We may note the following :—

(a) Put $\alpha = \frac{\pi}{4}$, then $\tan \alpha = 1$. We obtain the series of section 1. The properties of these co-efficients will be discussed in another paper. We observe that

$$A_n = n! + (n-2)! \sum (n-1)^2 + (n-4)! \sum (n-3)^2 \sum (n-2)^2 + \dots$$
$$+ (n-2r)! \sum (n-2r+1)^2 \sum (n-2r+2)^2 \dots \sum (n-r)^2 + \dots$$

(b) Put $\alpha = \frac{\pi}{3}$, $\tan \alpha = \sqrt{3}$ We easily find that, if the expansion be written in the form,

$$1 + b_1 x + b_2 \frac{x^2}{2!} + \dots + b_n \frac{x^n}{n!} + \dots,$$

$$\text{then} \quad 1-\frac{1}{5^{2n}}-\frac{1}{7^{2n}}+\frac{1}{11^{2n}}+\frac{1}{13^{2n}}-\ldots=\frac{\pi^{2n}}{6^{2n}}\,2.\,\frac{b_{2n-1}}{(2n-1)\,!} \tag{10}$$

$$1+\frac{1}{5^{2n+1}}-\frac{1}{7^{2n+1}}-\frac{1}{11^{2n+1}}+\ldots=\frac{\pi^{2n+1}}{6^{2n+1}}\,2.\,\frac{b_{2n}}{2n\,!}.$$

We find also that b_n is to be calculated from the equations,

$$\left.\begin{aligned} b_{2n} &= \binom{2n}{1} b_{2n-1}\sqrt{3}+\binom{2n}{2} b_{2n-2}-\ldots \\ &\qquad +(-1)^{n-1}\binom{2n}{2n-1} b_1\sqrt{3}+(-1)^{n-1}, \\ b_{2n+1} &= \binom{2n+1}{1} b_{2n}\sqrt{3}+\binom{2n+1}{2} b_{2n-1}-\ldots \\ &\quad +(-1)^{n-1}\binom{2n+1}{2n-1} b_2\sqrt{3}+(-1)^{n-1}\binom{2n+1}{2n} b+(-1)^{n}. \end{aligned}\right\} \tag{11}$$

The values of the first few co-efficients are,

$$b_1=\sqrt{3},\ b_2=7,\ b_3=23\sqrt{3},\ b_4=305,\ b_5=1685\sqrt{3},\ \ldots$$

(c) Put $\alpha=\frac{\pi}{6}$, then $\tan\alpha=\frac{1}{\sqrt{3}}$. If we write the expansion as

$$1+a_1x+a_2\frac{x^2}{2!}+\ldots+a_n\frac{x^n}{n!}+\ldots,$$

$$\left.\begin{aligned} &\text{then}\quad 1-\frac{1}{2^{2n}}-\frac{1}{4^{2n}}+\frac{1}{5^{2n}}+\frac{1}{7^{2n}}-\ldots=\frac{\pi^{2n}}{3^{2n}}\cdot\frac{2}{\sqrt{3}}\cdot\frac{a_{2n-1}}{(2n-1)\,!} \\ &1+\frac{1}{2^{2n+1}}-\frac{1}{4^{2n+1}}-\frac{1}{5^{2n+1}}+\frac{1}{7^{2n+1}}+\frac{1}{8^{2n-1}}-\ldots \\ &\qquad\qquad =\frac{\pi^{2n+1}}{3^{2n+1}}\,\frac{2}{\sqrt{3}}.\,\frac{a_{2n}}{2n\,!}. \end{aligned}\right\} \tag{12}$$

Also $\qquad a_1=\frac{1}{\sqrt{3}},\ a_2=\frac{5}{3},\ a_3=\frac{7}{\sqrt{3}},\ \ldots$

By comparing with the formulæ of Mr. Ranganathan (*loc. cit.*), we may write

$$\left.\begin{aligned} a_{2n} &= (-1)^{n+1}\cdot 3^{2n+1}\cdot\frac{(2^{2n+1}+1)}{2n+1}\cdot\phi_{2n+1}\left(\tfrac{1}{3}\right), \\ a_{2n+1} &= \frac{\sqrt{3}}{4(n+1)}\cdot(3^{2n+2}-1)\,(2^{2n+1}-1)\,B_n. \end{aligned}\right\} \tag{13}$$

(*d*) If $\alpha = \frac{\pi}{8}$, we get

$$\left.\begin{aligned} &\frac{1}{3^{2n}} - \frac{1}{5^{2n}} - \frac{1}{11^{2n}} + \frac{1}{13^{2n}} + \frac{1}{19^{2n}} - \dots\dots \\ &\qquad = \frac{\pi^{2n}}{4^{2n}} \frac{1}{2^{2n-1}} \frac{K_{2n-1}}{(2n-1)!} \frac{1}{\sqrt{2+\sqrt{2}}}, \\ &\frac{1}{3^{2n+1}} + \frac{1}{5^{2n+1}} - \frac{1}{11^{2n+1}} - \frac{1}{13^{2n+1}} + \dots \\ &\qquad = \frac{\pi^{2n+1}}{4^{2n+1}} \frac{1}{2^{2n-1}} \frac{K_{2n}}{2n\,!} \frac{1}{\sqrt{2+\sqrt{2}}}, \end{aligned}\right\} \quad (14)$$

where the K's are the corresponding co-efficients.

§ 3. Next consider the expansion of $\dfrac{\sin x + \tan\alpha \,.\cos x}{\cos x - \tan\alpha \,.\sin x}$ as an ascending power series in x. Write the expansion in the form,

$$\Sigma \,.\, L_n \,.\, \frac{x^n}{n!}. \qquad \dots \qquad \dots \qquad \dots \quad (15)$$

Now we have

$$\frac{\sin x + \tan\alpha \,.\cos x}{\cos x - \tan\alpha \,.\sin x} = \tan(x + \alpha)$$

$$= 2\sum_1^\infty \left\{ \frac{1}{(2n-1)\pi - 2(x+\alpha)} - \frac{1}{(2n-1)\pi + 2(x+\alpha)} \right\}$$

$$= 2\sum_1^\infty \left\{ \frac{1}{(2n-1)\pi - 2\alpha} \cdot \frac{1}{1 - \dfrac{2x}{(2n-1)\pi - 2\alpha}} - \frac{1}{(2n-1)\pi + 2\alpha} \cdot \frac{1}{1 + \dfrac{2x}{(2n-1)\pi + 2\alpha}} \right\}$$

$$= 2\left\{ \left(\frac{1}{\pi - 2\alpha} - \frac{1}{\pi + 2\alpha} + \frac{1}{3\pi - 2\alpha} - \frac{1}{3\pi + 2\alpha} + \dots \right) \right.$$

$$+ 2x\left[\frac{1}{(\pi - 2\alpha)^2} + \frac{1}{(\pi + 2\alpha)^2} + \frac{1}{(3\pi - 2\alpha)^2} + \frac{1}{(3\pi + 2\alpha)^2} + \dots \right]$$

$$+ 2^3x^3 \left[\frac{1}{(\pi - 2\alpha)^3} - \frac{1}{(\pi + 2\alpha)^3} + \frac{1}{(3\pi - 2\alpha)^3} - \frac{1}{(3\pi + 2\alpha)^3} + \cdots\right] + \cdots\cdots$$

$$+ 2^{2n-1}x^{2n-1} \left[\frac{1}{(\pi - 2\alpha)^{2n}} + \frac{1}{(\pi + 2\alpha)^{2n}} + \frac{1}{(3\pi - 2\alpha)^{2n}} + \frac{1}{(3\pi + 2\alpha)^{2n}} + \cdots\right]$$

$$+ 2^{2n}x^{2n} \left[\frac{1}{(\pi - 2\alpha)^{2n+1}} - \frac{1}{(\pi - 2\alpha)^{2n+1}} + \frac{1}{(3\pi - 2\alpha)^{2n+1}} - \frac{1}{(3\pi + 2\alpha)^{2n+1}} + \cdots\cdots\right] + \cdots\cdots \Bigg\} \quad \cdots \quad (16)$$

after re-arranging in powers of x.

Hence, by equating co-efficients, we obtain,

$$\left.\begin{aligned} \frac{L_{2n-1}}{(2n-1)!} &= 2 \cdot 2^{2n-1} \left[\frac{1}{(\pi - 2\alpha)^{2n}} + \frac{1}{(\pi + 2\alpha)^{2n}} + \frac{1}{(3\pi - 2\alpha)^{2n}} + \frac{1}{(3\pi + 2\alpha)^{2n}} + \cdots\cdots\right] \cdots \\ \frac{L_{2n}}{2n!} &= 2 \cdot 2^{2n} \left[\frac{1}{(\pi - 2\alpha)^{2n+1}} - \frac{1}{(\pi + 2\alpha)^{2n+1}} + \frac{1}{(3\pi - 2\alpha)^{2n+1}} - \frac{1}{(3\pi + 2\alpha)^{2n+1}} + \cdots\cdots\right]. \end{aligned}\right\} (17)$$

These equations are again fundamental, and by giving different values to α, various summations may be obtained. The following are interesting. The general expression for L_n as a function of $\tan\alpha$ has yet to be investigated.

(*a*) Put $\alpha = \frac{\pi}{4}$, then $\tan\alpha = 1$. The function to be expanded becomes

$$\frac{\sin x + \cos x}{\cos x - \sin x} = \frac{1 + \sin 2x}{\cos 2x} = \sec 2x + \tan 2x.$$

We obtain the well-known summations in terms of Euler's and Bernoulli's numbers.

(*b*) Put $\alpha = \frac{\pi}{3}$, $\tan\alpha = \sqrt{3}$. Hence if we write

$$\frac{\sin x + \sqrt{3}\cos x}{\cos x - \sqrt{3}\sin x} = \sqrt{3}\left\{1 + c_1 x + \frac{c_2}{2!} \cdot x^2 + \cdots + \frac{c^n}{n!} \cdot x^n + \cdots\right\},$$

it follows that,

$$\sqrt{3} \cdot \frac{c_{2n-1}}{(2n-1)!} \cdot \frac{\pi^{2n}}{6^{2n}} = 1 + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \frac{1}{11^{2n}} + \frac{1}{13^{2n}} + \dots\dots\dots$$

$$\sqrt{3} \cdot \frac{c_{2n}}{2n!} \cdot \frac{\pi^{2n+1}}{6^{2n+1}} = 1 - \frac{1}{5^{2n+1}} + \frac{1}{7^{2n+1}} - \frac{1}{11^{2n+1}} + \dots\dots\dots \qquad (18)$$

It will be seen that

$$c_1 = \frac{4}{\sqrt{3}},\ c_2 = 8,\ c_3 = \frac{80}{\sqrt{3}},\ c_4 = 352,\ c_5 = \frac{5824}{\sqrt{3}},\ c_6 = 38528, \dots\dots$$

and that

$$1 + \frac{1}{5^6} + \frac{1}{7^6} + \frac{1}{11^6} + \dots\dots = \frac{91\,\pi^6}{87480} \quad \dots \quad \dots \quad (19)$$

The result given by Mr. Ranganathan in his paper for the sum of this series seems to be incorrect. By reference to equations given in continuation of this paper, we have

$$c_{2n} = (-1)^n \cdot 2 \cdot 6^{2n} \cdot \psi_{2n}(\tfrac{1}{3}).$$

$$c_{2n-1} = \frac{1}{\sqrt{3}} \cdot \left\{ (-1)^n \psi_{2n-1}(\tfrac{1}{3}) + \frac{2^{2n}-1}{3^{2n}} \cdot \frac{B_n}{2n} \right\}$$

or

$$\frac{2^{2n-1}}{2n} \cdot (3^{2n}-1)\,(2^{2n}-1)\,\frac{B_n}{\sqrt{3}}$$

from Mr. Ranganathan's formula (*loc. cit.*)

(c) Put $\alpha = \frac{\pi}{6}$, $\tan\alpha = \frac{1}{\sqrt{3}}$. If we write the expansion as

$$\frac{1}{\sqrt{3}} \cdot \left\{ 1 + d_1 x + d_2 \frac{x^2}{2!} + \dots d_n \cdot \frac{x^n}{n!} + \dots \right\},$$

we find $\quad d_1 = \frac{4}{\sqrt{3}},\ d_2 = \frac{8}{3},\ d_3 = \frac{16}{\sqrt{3}},\ d_4 = 32, \dots \quad \dots \quad (20)$

$$d_{2n} = \binom{2n}{1} d_{2n-1} \cdot \frac{1}{\sqrt{3}} + \binom{2n}{2} d_{2n-2} - \binom{2n}{3} d_{2n-3} \frac{1}{\sqrt{3}} - \dots$$
$$+ (-1)^{n-2} \binom{2n}{2n-2} d_2 + (-1)^{n-1} \binom{2n}{2n-1} d_1 \cdot \frac{1}{\sqrt{3}}$$

$$d_{2n+1} = \binom{2n+1}{1} d_{2n} \cdot \frac{1}{\sqrt{3}} + \binom{2n+1}{2} d_{2n-1} - \dots$$
$$+ (-1)^{n-1} \binom{2n+1}{2n} d_1 + (-1)^n \cdot \frac{1}{\sqrt{3}} + (-1)^n \cdot \sqrt{3}$$

and also,

$$\left.\begin{aligned} \frac{1}{\sqrt{3}} \cdot \frac{d_{2n-1}}{(2n-1)!} \cdot \frac{\pi^{2n}}{3^{2n}} &= 1 + \frac{1}{2^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \dots\dots \\ \frac{1}{\sqrt{3}} \cdot \frac{d_{2n}}{2n!} \cdot \frac{\pi^{2n+1}}{3^{2n+1}} &= 1 - \frac{1}{2^{2n+1}} + \frac{1}{4^{2n+1}} - \frac{1}{5^{2n+1}} + \dots\dots \end{aligned}\right\} \quad (21)$$

(*d*) If $\alpha = \frac{\pi}{8}$, and l_n denote the corresponding co-efficient, we have

$$\left.\begin{aligned} \frac{l_{2n-1}}{(2n-1)!} \cdot \frac{\pi^{2n}}{8^{2n}} &= \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{11^{2n}} + \frac{1}{13^{2n}} + \ldots\ldots \\ \frac{l_{2n}}{2n!} \cdot \frac{\pi^{2n+1}}{8^{2n+1}} &= \frac{1}{3^{2n+1}} - \frac{1}{5^{2n+1}} + \frac{1}{11^{2n+1}} - \frac{1}{13^{2n+1}} + \ldots\ldots \end{aligned}\right\} \quad (22)$$

§ 4. The above formulæ may also be established by an application of Cauchy's Theorem of residues, as under :

Consider the expansion of the function $f(z) = \dfrac{1}{\cos z - \sin z}$ in ascending powers of z. Since $|\, z f(z)\,| \rightarrow 0$ uniformly as $z \rightarrow \infty$, it follows by Cauchy's theorem that the sum of the residues of the function at all its poles is zero. Its poles are given by $\tan z = 1$, that is,

$$z = \frac{\pi}{4},\ \frac{5\pi}{4},\ \frac{9\pi}{4} \ldots -\frac{3\pi}{4},\ -\frac{7\pi}{4},\ -\frac{11\pi}{4} \ldots$$

Let
$$\frac{1}{\cos z - \sin z} = 1 + \Sigma\, A_n \frac{z^n}{n!}$$

Then $\dfrac{A_{n-1}}{(n-1)!}$ is the residue at the origin of the function

$$\phi(z) = \frac{1}{z^n(\cos z - \sin z)}.$$

Residue of $\phi(z)$ at $\frac{\pi}{4} = \left[\frac{4}{\pi}\right]^n \cdot \frac{1}{-\sqrt{2}}$.

" " $\frac{5\pi}{4} = \left[\frac{4}{5\pi}\right]^n \cdot \frac{1}{\sqrt{2}}$

" " $\frac{9\pi}{4} = \left[\frac{4}{9\pi}\right]^n \cdot \frac{1}{-\sqrt{2}}$.

" " $-\frac{3\pi}{4} = \left[-\frac{4}{3\pi}\right]^n \cdot \frac{1}{\sqrt{2}}$.

" " $-\frac{7\pi}{4} = \left[-\frac{4}{7\pi}\right]^n \cdot \frac{1}{-\sqrt{2}}$. etc.

Hence
$$\frac{A_{n-1}}{(n-1)!} + \left[\frac{4}{\pi}\right]^n \cdot \frac{1}{\sqrt{2}} \cdot \left\{ -\frac{1}{1^n} + \frac{1}{5^n} - \frac{1}{9^n} + \ldots + (-1)^n \cdot \left(\frac{1}{3^n} - \frac{1}{7^n} + \frac{1}{11^n} - \ldots\right) \right\} = 0.$$

$$\therefore\ \frac{1}{1^n}-\frac{1}{5^n}+\frac{1}{9^n}-\dots+(-1)^{n+1}\left[\frac{1}{3^n}-\frac{1}{7^n}+\frac{1}{11^n}-\dots\right]$$

$$=\frac{A_{n-1}}{(n-1)!}\cdot\left[\frac{\pi}{4}\right]^n\sqrt{2}$$

Hence we have the formula established in § 1.

§ 5. *Application to Definite Integrals.*

We know that $\int_0^\infty e^{-ax}\,x^{n-1}\,dx=\frac{\Gamma(n)}{a^n}$.

Hence
$$\frac{1}{(\pi-2\alpha)^{2n}}-\frac{1}{(\pi+2\alpha)^{2n}}-\frac{1}{(3\pi-2\alpha)^{2n}}+\frac{1}{(3\pi+2\alpha)^{2n}}$$
$$+\frac{1}{(5\pi-2\alpha)^{2n}}-\frac{1}{(5\pi+2\alpha)^{2n}}-\frac{1}{(7\pi-2\alpha)^{2n}}+\dots\dots\dots$$
$$=\frac{1}{\Gamma(2n)}\int_0^\infty x^{2n-1}\left\{e^{-x(\pi-2\alpha)}-e^{-x(\pi+2\alpha)}-e^{-x(3\pi-2\alpha)}\right.$$
$$\left.+e^{-x(3\pi+2\alpha)}+e^{-x(5\pi-2\alpha)}-\dots\right\}dx.$$
$$=\frac{1}{\Gamma(2n)}\int_0^\infty x^{2n-1}\cdot\frac{e^{2\alpha x}-e^{-2\alpha x}}{e^{\pi x}+e^{-\pi x}}.\,dx.$$

Hence from §2, we obtain,

$$\frac{A_{2n-1}(\tan\alpha)}{(2n-1)!}\cdot\frac{1}{2^{2n}\cos\alpha}=\frac{1}{\Gamma(2n)}\int_0^\infty x^{2n-1}\frac{e^{2\alpha x}-e^{-2\alpha x}}{e^{\pi x}+e^{-\pi x}}\,dx.$$

$$\therefore\ \frac{A_{2n-1}(\tan\alpha)}{2^{2n}\cos\alpha}=\int_0^\infty x^{2n-1}\frac{e^{2\alpha x}-e^{-2\alpha x}}{e^{\pi x}+e^{-\pi x}}\,dx.\qquad(23)$$

Similarly we have,

$$\frac{A_{2n}(\tan\alpha)}{2^{2n+1}\cos\alpha}=\int_0^\infty x^{2n}\frac{e^{2\alpha x}+e^{-2\alpha x}}{e^{\pi x}+e^{-\pi x}}\,dx.\qquad(24)$$

It is to be remembered that A_n as a function of $a=\tan\alpha$, has been written down already in § 2. The following are examples :—

(a) $\alpha=\frac{\pi}{4}$. We have,

$$\frac{A_{2n-1}}{2^{2n}}\cdot\sqrt{2}=\int_0^\infty x^{2n-1}\cdot\frac{\left(e^{\frac{\pi x}{2}}-e^{-\frac{\pi x}{2}}\right)}{e^{\pi x}+e^{-\pi x}}\,dx,$$

which is transformed by the substitution $\frac{\pi x}{2} = y$ into

$$A_{2n-1}\left(\frac{\pi}{4}\right)^{2n}.\sqrt{2} = \int_0^\infty \frac{\sinh x}{\cosh 2x}.x^{2n-1}dx$$

$$= \int_0^1 \frac{1-x^2}{1+x^4}.\left(\log\frac{1}{x}\right)^{2n-1}dx \qquad [y=e^{-x} \qquad (25)$$

$$= \int^{\pi/2} \frac{\cos x}{1+\cos^2 x}\left(\log\cot\frac{x}{2}\right)^{2n-1}dx$$

[$x = \tan y$ in the last.

A_{2n-1} being the same as in § 1.

Similarly

$$\frac{A_{2n}}{2^{2n+1}}.\sqrt{2} = \int_0^\infty \frac{e^{\frac{\pi x}{2}} + e^{-\frac{\pi x}{2}}}{e^{\pi x} + e^{-\pi x}}.x^{2n}\,dx$$

which may be expressed as

$$A_{2n}\left(\frac{\pi}{4}\right)^{2n+1}.\sqrt{2} = \int_0^\infty \frac{\cosh x}{\cosh 2x}\,x^{2n}\,dx$$

$$= \int_0^1 \frac{1+y^2}{1+y^4}.\left(\log\frac{1}{y}\right)^{2n}dy$$

$$= \int_0^{\pi/2} \frac{\left(\log\cot\frac{x}{2}\right)^{2n}}{1+\cos^2 x}\,dx, \qquad \ldots \quad (26)$$

by the same substitutions as above. Since the values of A_n are known, we can write down particular integrals by giving different values to n.

(*b*) Put $\alpha = \frac{\pi}{3}$. We obtain from § 2 (*b*) that

$$\frac{b_{2n-1}}{(2n-1)!}.\frac{1}{2^{2n-1}} = \frac{1}{\Gamma(2n)}.\int_0^\infty x^{2n-1}.\frac{e^{\frac{2\pi}{3}x} - e^{-\frac{2\pi}{3}x}}{e^{\pi x} + e^{-\pi x}}.\,dx$$

which may be expressed as

$$b_{2n-1}.2.\left(\frac{\pi}{6}\right)^{2n} = \int_0^\infty \frac{\sinh 2x}{\cosh 3x}.x^{2n-1}\,dx$$

$$= \int_0^1 \frac{1-x^4}{1+x^6}.\left(\log\frac{1}{x}\right)^{2n-1}dx$$

$$= \int_0^{\pi/2} \frac{2\cos x}{1+3\cos^2 x}.\left(\log\cot\frac{x}{2}\right)^{2n-1}dx. \qquad (27)$$

Similarly,

$$b_{2n} \cdot 2 \cdot \left(\frac{\pi}{6}\right)^{2n+1} = \int_0^\infty \frac{\cosh 2x}{\cosh 3x} \cdot x^{2n}\, dx$$

$$= \int_0^1 \frac{1+x^4}{1+x^6} \cdot \left(\log \frac{1}{x}\right)^{2n} dx$$

$$= \int_0^{\pi/2} \frac{1+\cos^2 x}{1+3\cos^2 x} \cdot \left(\log \cot \frac{x}{2}\right)^{2n} dx. \qquad (28)$$

(c) Put $\alpha = \frac{\pi}{6}$. We obtain,

$$a_{2n-1} \cdot \left(\frac{\pi}{6}\right)^{2n} \cdot \frac{2}{\sqrt{3}} = \int_0^\infty \frac{\sinh x}{\cosh 3x} \cdot x^{2n-1}\, dx$$

$$= \int_0^1 \frac{1-y^2}{1+y^6} \cdot y \left(\log \frac{1}{y}\right)^{2n-1} dy$$

$$= \int_0^{\pi/2} \frac{\sin x \cos x}{1+3\cos^2 x} \cdot \left(\log \cot \frac{x}{2}\right)^{2n-1} dx. \qquad (29)$$

Similarly

$$a_{2n} \left(\frac{\pi}{6}\right)^{2n+1} \cdot \frac{2}{\sqrt{3}} = \int_0^\infty x^{2n} \cdot \frac{e^x + e^{-x}}{e^{3x} + e^{-3x}} \cdot dx.$$

$$= \int_0^\infty x^{2n} \cdot \frac{\cosh x}{\cosh 3x} \cdot dx.$$

$$= \int_0^1 \frac{1+x_{\circ}}{1+x^6} \cdot x \left(\log \frac{1}{x}\right)^{2n} dx. \qquad (30)$$

$$= \int_0^{\pi/2} \frac{\sin x}{1+3\cos^2 x} \cdot \left(\log \cot \frac{x}{2}\right)^{2n} dx.$$

(d) Put $\alpha = \frac{\pi}{m}$ generally. If X_n denotes the corresponding coefficient, we have

$$X_{2n-1} \left(\frac{\pi}{2m}\right)^{2n} \cdot \frac{1}{\cos\frac{\pi}{m}} = \int_0^\infty x^{2n-1} \frac{e^{2x} - e^{-2x}}{e^{mx} + e^{-mx}} \cdot dx. \qquad (31)$$

$$X_{2n} \cdot \left(\frac{\pi}{2m}\right)^{2n+1} \cdot \frac{1}{\cos\frac{\pi}{m}} = \int_0^\infty x^{2n} \cdot \frac{e^{2x} + e^{-2x}}{e^{mx} + e^{-mx}} \cdot dx. \qquad (32)$$

§ 6. We have again from § 3,

$$\frac{1}{(\pi-2\alpha)^{2n}}+\frac{1}{(\pi+2\alpha)^{2n}}+\frac{1}{(3\pi-2\alpha)^{2n}}+\frac{1}{(3\pi+2\alpha)^{2n}}$$
$$+\frac{1}{(5\pi-2\alpha)^{2n}}+\frac{1}{(5\pi+2\alpha)^{2n}}+\dots\dots$$
$$=\frac{1}{\Gamma(2n)}\int_0^\infty x^{2n-1}\Big\{e^{-x(\pi-2\alpha)}+e^{-x(\pi+2\alpha)}$$
$$+e^{-x(3\pi-2\alpha)}+e^{-x(3\pi+2\alpha)}+\dots\Big\}dx$$
$$=\frac{1}{\Gamma(2n)}\int_0^\infty x^{2n-1}\cdot\frac{e^{2\alpha x}+e^{-2\alpha x}}{e^{\pi x}-e^{-\pi x}}\,dx.$$

Hence we obtain,

$$\frac{L_{2n-1}}{2^{2n}}=\int_0^\infty x^{2n-1}\cdot\frac{e^{2\alpha x}+e^{-2\alpha x}}{e^{\pi x}-e^{-\pi x}}\,dx. \tag{33}$$

Similarly,

$$\frac{L_{2n}}{2^{2n+1}}=\int_0^\infty x^{2n}\frac{e^{2\alpha x}-e^{-2\alpha x}}{e^{\pi x}-e^{-\pi x}}\,dx. \tag{34}$$

These formulae are general, and we can obtain particular cases as follows :

(*a*) Put $\alpha=\frac{\pi}{4}$. We get the familiar integrals for B_n and E_n. We need not dilate on these.

(*b*) Put $\alpha=\frac{\pi}{3}$. We obtain,

$$\sqrt{3}.\ \frac{c_{2n-1}}{(2n-1)!}\cdot\frac{1}{2^{2n}}\ .=\frac{1}{\Gamma(2n)}\int_0^\infty x^{2n-1}\frac{e^{\frac{2\pi x}{3}}+e^{-\frac{2\pi x}{3}}}{e^{\pi x}-e^{-\pi x}}dx \tag{35}$$

or what is the same thing,

$$\sqrt{3}\ .\ c_{2n-1}\cdot\left(\frac{\pi}{6}\right)^{2n}=\int_0^\infty x^{2n-1}\cdot\frac{e^{2x}+e^{-2x}}{e^{3x}-e^{-3x}}\,dx$$
$$=\int_0^1\frac{1+x^4}{1-x^6}\cdot\left(\log\frac{1}{x}\right)^{2n-1}dx$$
$$=\int_0^{\pi/2}\frac{1+\cos^2 x}{\cos x(3+\cos^2 x)}\cdot\left(\log\cot\frac{x}{2}\right)^{2n-1}dx. \tag{36}$$

Similarly,

$$\sqrt{3}\,.\,c_{2n}\,.\,\left(\frac{\pi}{6}\right)^{2n+1} = \int_0^\infty x^{2n}\,.\,\frac{e^{2x}-e^{-2x}}{e^{3x}-e^{-3x}}\,dx. \qquad (37)$$

(c) Put $\alpha = \frac{\pi}{6}$. We obtain from § 3, (c) that

$$\frac{1}{\sqrt{3}}\,.\,d_{2n-1}\,.\,\left(\frac{\pi}{6}\right)^{2n} = \int_0^\infty x^{2n-1}\,.\,\frac{e^{x}+e^{-x}}{e^{3x}-e^{-3x}}\,dx.$$

$$\frac{1}{\sqrt{3}}\,.\,d_{2n}\,.\,\left(\frac{\pi}{6}\right)^{2n+1} = \int_0^\infty x^{2n}\,.\,\frac{e^{x}-e^{-x}}{e^{3x}-e^{-3x}}\,dx. \qquad (38)$$

(d) Put $\alpha = \frac{\pi}{m}$ generally. Then if γ_n denotes the corresponding co-efficient, we have

$$\frac{\gamma_{2n-1}}{2^{2n}}\,.\,\left(\frac{\pi}{m}\right)^{2n} = \int_0^\infty x^{2n-1}\,.\,\frac{e^{2x}+e^{-2x}}{e^{mx}-e^{-mx}}\,dx.$$

$$\frac{\gamma_{2n}}{2^{2n+1}}\,.\,\left(\frac{\pi}{m}\right)^{2n+1} = \int_0^\infty x^{2n}\,.\,\frac{e^{2x}-e^{-2x}}{e^{mx}-e^{-mx}}\,dx. \qquad \ldots \quad (39)$$

§ 7. The following are a few additional integrals. Substituting for the series in § 2 the definite integrals obtained in the last section, we have

$$\sec(x+\alpha) = 2\left[\Sigma\,\frac{2^{2n-1}x^{2n-1}}{(2n-1)\,!}\int_0^\infty z^{2n-1}\,\frac{e^{2\alpha z}-e^{-2\alpha z}}{e^{\pi z}+e^{-\pi z}}\,dz + \Sigma\,\frac{2^{2n}x^{2n}}{2n\,!}\int_0^\infty z^{2n}\,\frac{e^{2\alpha z}+e^{-2\alpha z}}{e^{\pi z}+e^{-\pi z}}\,dz\right]$$

$$= 2\int_0^\infty \frac{\sinh 2xz\,.\,\sinh 2\alpha z}{\cosh \pi z}\,dz + 2\int_0^\infty \frac{\cosh 2xz\,.\,\cosh 2\alpha z}{\cosh \pi z}\,dz.$$

$$\therefore\ \sec(x+\alpha) = 2\int_0^\infty \frac{\cosh (x+\alpha)\,2z}{\cosh \pi z}\,dz. \qquad \ldots \quad (40)$$

This is a well-known integral for sec x. Similarly from § 3, and § 4, we obtain the well-known integral for tan x.

N.B.—[The co-efficients A_n introduced in this Note have interesting properties similar to Euler's and Bernoulli's numbers. These will be discussed in a future paper.]

SHORT NOTES.

Linear Systems of the third Order on a Conic.

[Reference: 'Linear Systems of Points,' *J. I. M. S*, June 1919].

1. A general involution relation on a conic might be visualised as the relation between the extremities of a chord passing through a fixed point in the plane. But pairs of points in involution form a particular case of linear systems and there arises the question of representation on the conic of linear systems of higher order, especially the third. There are two types of systems of the third order—the complete and the incomplete. The former consists of ∞^2 triads (or in the geometrical representation of ∞^2 inscribed triangles) and the latter, of ∞ triads.

2. *Representation of the incomplete system or the pencil of inscribed triangles.*

The incomplete system of the third order (*i.e.*, as we have termed it, the pencil of inscribed triangles) corresponds to a straight line in three dimensions and is thus determined by any two of its members. Now let $A_1B_1C_1$, $A_2B_2C_2$ be two triangles inscribed in the fundamental conic S and let S′ be the unique conic inscribed to each of the triangles. Then it is known that there are an infinity of triangles inscribed in and circumscribed to S, S′ respectively, and if *any one* vertex of any of these triangles is known the triangle itself is determined uniquely; hence the set of these triangles constitutes the pencil determined by $A_1B_1C_1$, $A_2B_2C_2$.

It might be shewn similarly that the set of triangles inscribed in S so as to be self-conjugate to another conic form a pencil.

Null Pencils. If the pencil is a *focal* pencil, S′ must have double contact with S.

Since every group of the third order is self-harmonic, it follows that if *two* triangles of a pencil are harmonic, then *any two* triangles of the pencil are harmonic. Such a pencil will be called a *null pencil.*

Since a pencil corresponds to a straight line in the representative three-space with a fundamental twisted cubic,* null pencils should be represented by lines specially related to the cubic; in fact by the lines of the linear complex determined by the cubic. A null pencil containing a given triad p should be represented by a straight line through the

* The standard representation of linear systems as flat spaces in higher dimensions with a fundamental normal curve is dealt with in the paper mentioned in the title.

corresponding point P lying wholly in the polar plane of P and *therefore* meeting the unique line-in-two-planes contained in that polar plane. Hence the null pencil should be the pencil determined by p and a triad of the form $(\alpha\beta\gamma)$ where $\beta\gamma$ are the indeterminate pair (Hessian points) of p and α may be arbitrary. Hence the null pencil containing a given inscribed triangle of S is the set of triangles inscribed in S and circumscribed to S′ where S′ may be any conic, inscribed in the triangle and *touching the polar line of the triangle* with respect to S′; that is, S′ is a conic such that the invariants θ, θ' of S and S′ both vanish; or, as we might agree to say, S′ is *harmonic* to S. We thus reach the theorem:

The null pencil of inscribed triangles of S is the set of triangles circumscribed to (or self-conjugate to) a harmonic conic of S.

The Harmonic Relation. Two inscribed triangles Δ_1, Δ_2 are harmonic when the pencil which they determine is a null pencil; hence

Two inscribed triangles Δ_1, Δ_2 of S are harmonic when the conic inscribed to Δ_1 and Δ_2 is harmonic to S.

3. *First representation of the Complete System.* The geometrical meaning of the harmonic relation obtained above leads immediately to a representation of a complete system. The complete system whose foci are the points ABC on S is the set of all triangles harmonic to ABC, that is, the set of all triangles inscribed in S and circumscribed to the various conics of the 4-line system determined by the sides of ABC and the polar line of ABC with respect to S.

To represent a complete system with a double focus.

Let the foci be ABB and let the tangents at A, B to S meet in T. Then the complete system consists of triangles inscribed in S and circumscribed to the conics touching AB at A and TB at T.

4. *Second representation of the Complete System.* Let P be a point on S and Q, R points not on S. Then the system of conics passing through PQR intersect S again in triads which evidently belong to a complete system. If QR intersects S in UU′, then it is obvious from elementary geometry that

(1) UU′ is the indeterminate pair of the complete system.

(2) If Q′R′ be two points on QR such that QR, Q′R′, UU′ belong to an involution, then the complete system determined on S in the above manner by PQ′R′ is exactly the same as the system determined by PQR.

Let ABC be the foci of the complete system determined as above by PQR. Then the line QR should be the polar line of ABC with respect

to S. Let PA, PB, PC and the tangents at ABC cut QR in $A_1B_1C_1$, $A_2B_2C_2$, respectively. Then, by elementary geometry and the fact that A, B, C are foci of the system, it follows that the involution determined by QR and UU′ contains the pairs A_1A_2, B_1B_2, C_1C_2.

Hence in representing a *given* complete system with foci ABC in this manner, the point P is entirely arbitrary, the line QR is determined by ABC only, the points QR are unspecified except in that they belong to an involution which always contains UU′ and which is further completely determined when P is chosen.

We can shew that no material alteration in the terms of our representation is necessary when the system has a double focus.

5. *The complete system determined by three inscribed triangles* Δ_1, Δ_2, Δ_3.

Let S_1, S_2, S_3 be the conics inscribed to every two of these triangles and let L_1, L_2, L_3 be the polar lines of these triangles with respect to S. There is a fourth common tangent to S_2 and S_3 besides the sides of Δ_1; call this fourth common tangent L. The intersections of L with S have one correspondent in the pencil (Δ_1, Δ_2) and another correspondent in the pencil (Δ_1, Δ_3). Hence these intersections have more than one correspondent in the complete system $(\Delta_1\ \Delta_2\ \Delta_3)$ and therefore must form the *indeterminate pair* of the complete system. We thus have the theorm :—

The fourth common tangent to any two of the conics S_1, S_2, S_3 is the same line L.

Further, if Δ be the focal triangle, the pencil (Δ, Δ') is a null pencil, hence the conic inscribed to Δ and Δ_1 must touch L and L_1 (since these are the polar lines of Δ and Δ_1). Hence if P_r be the conic touching L, L_r and the sides of Δ_r, we have the theorem :

The conics P_1 P_2 P_3 belong to a four-line system. Their common tangents other than L *form a triangle which is inscribed in S and which is, in fact, the focal triangle of the complete system* $(\Delta_1\ \Delta_2\ \Delta_3)$.

6. *The Parabola.*

The feet of concurrent normals is a complete system of the third order whose foci are A ∞ ∞ (where A is the vertex and ∞, the point at infinity on the parabola). We call this the *co-normal system.* The triangle formed by the feet of concurrent normals will be termed 'a co-normal triangle.' The triads of points the centres of curvature at which are collinear form another complete system with the foci AA ∞. This we call the *co-central system.*

From the first representation of complete systems, we have—

(1) Any co-normal triangle is circumscribed to a parabola with the same vertex A and with its axis perpendicular to the axis of the given parabola.

(2) Any co-central triangle is circumscribed to a rectangular hyperbola the asymptotes of which are the tangent at the vertex and the axis.

The second representation gives

(3) If a conic with definite asymptotic directions circumscribes a co-normal triangle, it cuts the parabola again in a fixed point P, such that the involution determined at ∞ by the asymptotic directions and AP and the tangent at P has the point at ∞ on the axis for a fixed point.

In particular, the circum-circles of co-normal triangles pass through the vertex.

More generally, a conic circumscribed to a co-normal triangle with one of its axes parallel to the axis of the parabola, necessarily passes through the vertex. In particular, the *other* parabola which passes through the vertex and is circumscribed to a co-normal triangle has its axis perpendicular to the axis of the given parabola.

(4) A conic which has an asymptote parallel to the axis and is circumscribed to a co-central triangle intercepts on the tangent at the vertex a length which is bisected at the vertex.

Any conic having the tangent at the vertex for an asymptote and circumscribing a co-central triangle is a rectangular hyperbola.

We may also prove the following :

(5) A co-normal triangle may also be in addition co-central. This happens when the point of concurrence of the normals lies on the perpendicular to the axis at the cusp of the evolute ; further in this case the line of collinearity of the centres of curvature is a *diameter* of the parabola. The axis of the parabola is midway between the point of concurrence and the line of collinearity.

The polar line of a co-normal co-central triangle with respect to the parabola is the axis.

(6) Two co-normal triangles are harmonic when the join of their points of concurrence is a diameter ; two co-central triangles are harmonic when their lines of collinearity meet on the perpendicular to the axis at the cusp of the evolute. It is also easy to find the relation between the point of concurrence and the line of collinearity of two harmonic triangles one of which is co-normal and the other co-central.

R. Vaidyanathaswami.

SOLUTIONS.

Question 754.

(S. RAMANUJAN) :—Show that

$$\frac{e^x \Gamma(1+x)}{x^x \sqrt{\pi}} = \sqrt[6]{8x^3 + 4x^2 + x + E}$$

where E lies between $\frac{1}{100}$ and $\frac{1}{30}$ for all positive values of x.

Remarks by E. H. Neville and C. Krishnamachary.

Mr. Madhava's method (vol. xii, pp. 101-102) leads much nearer to the solution than he supposes, for in passing from the first formula on p. 102 to the third he has overlooked the alternations of sign on the left-hand side. On making the correction, we have

$$E = \tfrac{1}{30}, \quad F = -\tfrac{11}{240}.$$

With these values, Mr. Ramanujam's assertion is seen to be credible, but more powerful means must be used if it is to be proved.

Questions 908 and 1012.

(K. APPUKUTTAN ERADY, M.A.) :—Show that $(a_0 a_1 \ldots a_{2n})(x, y)^{2n}$ is reducible to the form

$$p_1(x + b_1 y)^{2n} + p_2(x + b_2 y)^{2n} + \ldots\ldots + p_n(x + b_n y)^{2n}$$

if

$$\begin{vmatrix} a_0 & a_1 & \ldots & \ldots & a_n \\ a_1 & a_2 & \ldots & \ldots & a_{n+1} \\ \ldots & & \ldots & \ldots & \ldots \\ \ldots & & \ldots & \ldots & \ldots \\ a_n & a_{n+1} & \ldots & \ldots & a_{2n} \end{vmatrix} = 0.$$

(N. SANKARA AIYAR) :—Solve

(1) $x^6 + 12x^5 + 45x^4 + 100x^3 + 120x^2 + 78x + 21 = 0;$

(2) $2x^4 + 12x^3 + 30x^2 + 32x + 12 = 0$

by expressing the left-hand members in the forms $A^6 + B^6$ and $A^4 + B^4$ respectively.

Additional solution by N. Sankara Aiyar.

Expanding $\Sigma p_r(x + b_r y)^{2n}$ and equating coefficients of x, we get

$$\Sigma p_r = a_0, \quad \Sigma p_r b_r = a_1, \quad \Sigma p_r b_r^k = a_k \ \ldots\ldots$$

the number of these equations being $(2n + 1)$.

Eliminating $p_1 p_2 \ldots\ldots p_n$ from the first $n+1$ of them, we get an equation giving $b_1 b_2 \ldots\ldots b_n$, viz ,

$$\begin{vmatrix} 1 & 1 & \ldots & \ldots & a_0 \\ b_1 & b_2 & \ldots & \ldots & a_1 \\ b_1^2 & b_2^2 & \ldots & \ldots & a_2 \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ b_1^n & b_2^n & \ldots & \ldots & a_n \end{vmatrix} = 0.$$

This can be expanded and written in the form

$\sum a_r A_r = 0$ where A_r is the minor of a_r.

Again from the second $n+1$ equations we shall get

$$\begin{vmatrix} b_1 & b_2 & \ldots & \ldots & a_1 \\ b_1^2 & b_2^2 & \ldots & \ldots & a_2 \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ b_1^{n+1} & b_2^{n+1} & \ldots & \ldots & a_{n+1} \end{vmatrix} = 0.$$

which may be written in the form $\sum a_{r+1} A_r = 0$.

From the $n+1$ equations of the type $\sum a_{r+k} A_r = 0$ we get by eliminating the minors the required relation between the a's. If this relation is satisfied the $(n+1)$ equations are consistent and we get the values of A_r and hence by solving the equations $A_r = k$ we can get the values of the b's and lastly of the p's. As an example let us take the function $x^4 + 8x^3 + 18x^2 + 20x + 8$, where the coefficients are easily seen to satisfy the equation.

Hence the equations for b_1, b_2 are

$$2b_1 b_2 - 3(b_1 + b_2) + 5 = 0$$
$$b_1 b_2 - 2(b_1 + b_2) + 3 = 0.$$

Since $b_1 \neq b_2$ this gives $b_1 = \dfrac{\sqrt{5} + 1}{2}$ and $b_2 = \dfrac{1 - \sqrt{5}}{2}$.

Substituting in the equations for the p's

we get $p_1 = \dfrac{5 - \sqrt{5}}{2}$ and $p_2 = \dfrac{\sqrt{5} - 3}{2}$.

Hence the given expression can be written as

$$\frac{5 - \sqrt{5}}{2}\left(x + \frac{\sqrt{5} + 1}{2}\right)^4 + \frac{\sqrt{5} - 3}{2}\left(x + \frac{1 - \sqrt{5}}{2}\right)^4.$$

Solution to Q. 1012, by K. B. Madhava.

[NOTE.—Applying this method to Q. 1012, it is easily seen that the polynomial expressions there given are equivalent to

$$\frac{5+3\sqrt{5}}{10}\left(x+\frac{\sqrt{5}+1}{2}\right)^6+\frac{5-3\sqrt{5}}{10}\left(x+\frac{1-\sqrt{5}}{2}\right)^6=0,$$

$$\frac{5+2\sqrt{5}}{5}\left(x+\frac{\sqrt{5}+1}{2}\right)^4+\frac{5-2\sqrt{5}}{5}\left(x+\frac{1-\sqrt{5}}{2}\right)^4=0;$$

from which the solutions of these equations are obvious.

In general, if the polynomial is of the form

$$x^{2n}+2nc_1 \,.\, A_1\, x^{2n-1}+2nc_2 \,.\, A_2\, x^{2n-2}+\ldots\ldots+A_{2n}=0,$$

where $\qquad A_{r+1}=A_r+A_{r-1}$ with $A_1=2$ and $A_2=3$;

it can be expressed in the form

$$\frac{5+3\sqrt{5}}{10}\left(x+\frac{\sqrt{5}+1}{2}\right)^{2n}+\frac{5-3\sqrt{5}}{10}\left(x+\frac{1-\sqrt{5}}{2}\right)^{2n}=0,$$

and the solution of the equation in this form is immediate.]

Question 957.

(MARTYN M. THOMAS, M.A.):—From a flexible envelope in the form of a surface of revolution formed by the curve $s=f(y)$ revolving about the axis of x, that part between two meridians the planes of which are inclined to each other at an angle $\frac{2\pi}{m}$ is cut away, and the edges are then sewed together. Prove that the meridian curve of the new envelope will be $s=f\left(\frac{my}{m-1}\right)$.

Hence show that, if a lune of angle $\frac{4\pi}{5}$ be cut off from an oblate spheroid, the minor axis of whose generating ellipse is c, and eccentricity $\frac{3}{5}$, the meridian curve of the new surface of revolution will be the curve of sines $y=\frac{3}{4}c\sin\frac{x}{c}$.

Solution by N. Sundaram Aiyar.

Let y be the radius of the circular section at distance x from the origin, and y' that of the new section at the same distance from the origin. The two perimeters of the section are in the ratio

$$\frac{m}{m-1}, \text{ so that } \frac{2\pi y}{2\pi y'}=\frac{m}{m-1}.$$

But the surface of revolution is formed by the curve $s = f(y)$. So, the meridian curve of the new envelope is $s = f\left(\frac{my}{m-1}\right)$.

The equation to the meridian curve of the oblate spheroid is $\frac{x^2}{c^2} + \frac{y^2}{a^2} = 1$ where a is the semi-major axis, so that $a^2 - c^2 = a^2 \cdot \frac{9}{25}$;

or $a = \frac{5c}{4}$; also $m = \frac{5}{2}$.

The equation to the curve is therefore $25x^2 + 16y^2 = 25c^2$.

$$\therefore \quad \frac{dx}{dy} = -\frac{16y}{25x} \text{ or } \left(\frac{ds}{dy}\right)^2 = 1 + \frac{256y^2}{25(25c^2 - 16y^2)}.$$

$$\therefore \quad \frac{ds}{dy} = \phi(y) = \sqrt{\frac{625c^2 - 144y^2}{25(25c^2 - 16y^2)}}.$$

$$\therefore \quad s = \int_0^y \phi(y)\, dy = \lambda(y) \text{ say.}$$

Then the meridian curve of the new surface of revolution is given by

$$s = \lambda\left(\frac{my}{m-1}\right) = \lambda\left(\frac{5y}{3}\right),$$

or

$$\frac{ds}{dy} = \frac{5}{3}\phi\left(\frac{5y}{3}\right) = \frac{5}{3}\sqrt{\frac{625c^2 - 16.25y^2}{25(25c^2 - 16.25y^2/9)}}$$

$$= \sqrt{\frac{25c^2 - 16y^2}{9c^2 - 16y^2}}.$$

$$\therefore \quad \left(\frac{dx}{dy}\right)^2 = \frac{16c^2}{9c^2 - 16y^2}$$

or

$$x = \int_0^y \frac{4c\,dy}{\sqrt{9c^2 - 16y^2}}$$

$$= c \sin^{-1}\frac{4y}{3c}$$

or

$$y = \frac{3c}{4}\sin\frac{x}{c}.$$

Question 996.

Find the conditions that the conics

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

and

$$a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0$$

have one asymptote in common.

Solution by P. R. Venkatakrishna Iyer and others.

Let $y = mx + n$ be the common asymptote. The x-co-ordinate of the points of intersection of $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and $y = mx + n$ are given by

$$ax^2 + 2hx(mx + n) + b(mx + n)^2 + 2gx + 2f(mx + n) + c = 0,$$

i.e., $x^2(a + 2hm + bm^2) + 2x(hn + bmn + g + fm) + bn^2 + 2fn + c = 0.$

If $y = mx + n$, is an asymptote, then

$$a + 2hm + bm^2 = 0, \quad \ldots \quad \ldots \quad \text{(i)}$$

and

$$hn + bmn + g + fm = 0. \quad \ldots \quad \ldots \quad \text{(ii)}$$

Similarly, if $y = mx + n$ is an asymptote of the other conic, then

$$a' + 2h'm + b'm^2 = 0, \quad \ldots \quad \ldots \quad \text{(iii)}$$

$$h'n + b'mn + g' + f'm = 0. \quad \ldots \quad \ldots \quad \text{(iv)}$$

Eliminating m between (i) and (iii), we get

$$H^2 = 4A.B, \quad \ldots \quad \ldots \quad \text{(I)}$$

where $A = bh' - b'h$; $B = ha' - h'a$; $H = ab' - a'b$.

From (ii) and (iv)

$$n = -\frac{g + fm}{h + bm} = -\frac{g' + f'm}{h' + b'm},$$

i.e., $m^2(b'f - bf') + m(b'g - bg' + h'f - hf') + (h'g - hg') = 0. \quad$ (v)

Eliminating m^2 and m between (ii), (iii) and (v), we have

$$\begin{vmatrix} a & 2h & b \\ a' & 2h' & b' \\ (h'g - hg') & (b'g - bg' + h'f - hf') & (h'g - hg') \end{vmatrix} = 0,$$

which on simplification reduces to

$$(g\sqrt{A} + f\sqrt{B})(b'\sqrt{B} + h'\sqrt{A}) = (g'\sqrt{A} + f'\sqrt{B})(b\sqrt{B} + h\sqrt{A}). \quad \text{(11)}$$

The conditions (I) and (II) are the required conditions for a common asymptote.

Question 1010.

(K. J. SANJANA, M.A.) :—Solve the equation $x^3 + y^3 = a^3 + b^3$, where a and b are given positive quantities and x and y are required to be positive.

Examples. $x^3 + y^3 = (\frac{5}{2})^3 + (\frac{3}{2})^3$, $x^3 + y^3 = 91 = 4^3 + 3^3$.

Solution by N. B. Mitra.

A particular solution of the problem may be given by following Vieta's method.

Let $$^3u - v^3 = a^3 + b^3.$$

Assume $$u = a + z,\ v = kz - b.$$

Then $$z^2(1 - k^3) + 3z(a + bk^2) + 3(a^2 - b^2k) = 0$$

To get a particular solution assume $k = a^2/b^2$.

Then $$z = \frac{3ab^3}{a^3 - b^3}, \quad \therefore\ u = \frac{a(a^3 + 2b^3)}{a^3 - b^3},\ v = \frac{b(2a^3 - b^3)}{a^3 - b^3}.$$

Now we have $$x^3 + y^3 = u^3 - v^3.$$

If $u^3 > 2v^3$, assume $x = p - v$, $y = u - rp$;
then, $p^3(1 - r^3) + 3p^2(r^2u - v) + 3p(v^2 - u^2r) = 0$.

A particular solution is obtained by putting $r = v^2/u^2$.

We get
$$p = \frac{3vu^3}{u^3 + v^3}$$
and finally
$$x = \frac{v(2u^3 - v^3)}{u^3 + v^3},\ y = \frac{u(u^3 - 2v^3)}{u^3 + v^3}.$$

If, however, $u^3 < 2v^3$, we proceed thus:

Assume $$u_1^3 - v_1^3 = u^3 - v^3.$$
Let $u_1 = p_1 - v$ and $v_1 = r_1p_1 - u$; then proceeding in the same way, we get
$$u_1 = \frac{v(2u^3 - v^3)}{u^3 + v^3},\ v_1 = \frac{u(2v^3 - u^3)}{u^3 + v^3};$$
and finally $$x = \frac{v_1(2u_1^3 - v_1^3)}{u_1^3 + v_1^3} \text{ and } y = \frac{u_1(u_1^3 - 2v_1^3)}{u_1^3 + v_1^3}.$$

Examples: (1) $a = \frac{5}{2}$, $b = \frac{3}{2}$: then
$$u = \frac{895}{196},\ v = \frac{669}{196},$$
and $$x = \frac{1134416441}{196 \, . \, 1016335684},\ y = \frac{895 \, . \, 118080757}{196 \, . \, 1016335684}.$$

(2) $a = 4$, $b = 3$: then
$$u = \frac{472}{37},\ v = \frac{303}{37},$$
and $$x = \frac{303 \, . \, 182489969}{132972175 \, . \, 37},\ y = \frac{472 \, . \, 49517794}{132972175 \, . \, 37}.$$

Question 1030.

(SELECTED) :—If $x^{2n-1} + y^{2n-1} + a_1^{2n-1} + a_2^{2n-1} + \dots + a_{2r}^{2n-1} = 0$, for all integral values of n from 1 to r inclusive, then

$$(x+a_1)(x+a_2)\dots\dots(x+a_{2r}) = (y+a_1)(y+a_2)\dots\dots(y+a_{2r}).$$

Solution by F. H. V. Gulasekharam, and N. G. Leather.

Let S_m denote the sum of the mth powers, and p_m, the sum of the m—ary products of the quantities $x, y, a_1, a_2, \dots\dots\dots\dots a_{2r}$. Let P_m denote the sum of the m —ary products of the quantities $a_1, a_2 \dots\dots\dots a_{2r}$.

By the question $\quad S_1 = S_3 = \dots\dots\dots\dots\dots\dots = S^{2r-1} = 0.$

Hence $\quad p_1 = p_3 = \dots\dots\dots\dots\dots\dots = p_{2r-1} = 0.$

These are equivalent to the following $(r-1)$ equations :—

$$(x+y) + p_1 = 0$$

$$p_1 \,.\, xy + p_2 \,.\, (x+y) + p_3 = 0$$

$$p_3 \,.\, xy + p_4 \,.\, (x+y) + p_5 = 0$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$p_{2r-3} \,.\, xy + p_{2r-2} \,.\, (x+y) + p_{2r-1} = 0.$$

Now, denoting $x^m - y^m$ by D_m, multiply the above $(r-1)$ equations in order by $D_{2r}, D_{2r-2}, \dots\dots\dots D_2$; and add the results. Then

$$D_{2r} \,.\, (x+y) + p_1 [D_{2r} + xy \,.\, D_{2r-2}]$$
$$+ p_2 D_{2r-2}(x+y) + p_3 [D_{2r-2} + xy\, D_{2r-4}] + \dots\dots = 0 \dots \quad (1)$$

Now remembering that $\quad D_{2m} + xy \,.\, D_{2m-2} \equiv (x+y) D_{2m-1}, \dots \quad (2)$

the equation (1) may be written in the form

$$(x+y)[D_{2r} + p_1 D_{2r-1} + p_2 D_{2r-2} + \dots\dots + p_{2r-2} D_2 + p_{2r-1} D_1] = 0$$

Hence,

$$x^{2r} + p_1 x^{2r-1} + \dots\dots\dots + p_{2r-1} x$$
$$= y^{2r} + p_1 y^{2r-1} \dots + p_{2r-1} y.$$

$$\therefore \quad (x+a_1)(x+a_2)\dots\dots(x+a_2{}^r)$$
$$= (y+a_1)\ (y+a_2)\dots\dots\dots(y+a_{2r}).$$

Question 1031.

(C. KRISHNAMACHARI) :—With the usual notation in elliptic functions, show that if $u + v + w = 0$, then

$$p(u)\,p(v)\,p(w) = \tfrac{1}{4} g_3 + \tfrac{1}{4} \left\{ \frac{p'(u)\,p(v) - p'(v)\,p(u)}{p(u) - p(v)} \right\}^2$$

Deduce that

$$p(z)\,[p(z+w_1)+p(z+w_2)+p(z+w_3)]$$

$$=\tfrac{1}{4}\,p'(z)\left\{\frac{1}{e_1-p(z)}+\frac{1}{e_2-p(z)}+\frac{1}{e_3-p(z)}\right\}.$$

Solution by F. H. V. Gulasekharam ; and V. Tiruvenkatachari.

Since $u+v+w=0$,

the points $\{p(u), p'(u)\}$, $\{p(v), p'(v)\}$, $\{p(w), p'(w)\}$,

of the curve $y^2=4x^3-g_2x-g_3$ are in a straight line.

Let the straight line be $y=\mathrm{A}x+\mathrm{B}$.

$$\therefore\quad \mathrm{A}\,p(u)+\mathrm{B}=p'(u);\ \mathrm{A}\,p(v)+\mathrm{B}=p'(v).$$

$$\therefore\quad \mathrm{B}=-\frac{p'(u)\,p(v)-p'(v)\,p(u)}{p(u)-p(v)},$$

also $p(u)$, $p(v)$, $p(w)$, are the roots of the cubic equation

$$4x^3-g_2x-g_3-(\mathrm{A}x+\mathrm{B})^2=0.$$

$$\therefore\quad p(u)\,p(v)\,p(w)=\tfrac{1}{4}g_3+\tfrac{1}{4}\mathrm{B}^2$$

$$=\tfrac{1}{4}g_3+\tfrac{1}{4}\left\{\frac{p'(u)\,p(v)-p(u)\,p'(v)}{p(u)-p(v)}\right\}^2.$$

The second part of the question seems to be inaccurate.

For, $\quad p(z)\,p(z+w_1)\,e_1=\tfrac{1}{4}g_3+\tfrac{1}{4}e_1{}^2.\dfrac{p'^2(z)}{(p(z)-e^1)^2}.$

Hence $p(z)\,\{p(z+w_1)+p(z+w_2)+p(z+w_3)\}+\tfrac{1}{4}g_2$ (after some easy reduction) is equal to

$$\frac{1}{4}\,p'^2(z)\,.\sum\frac{e_r}{\{p(z)-e_r\}^2}$$

[NOTE.—That the question is inaccurate can be easily seen by applying Liouville's Theorem ; for, a pole, say $z=0$, is of order 2 for the left side, while it is only simple for the right-hand expression : K. B. M.]

QUESTIONS FOR SOLUTION.

1179. (F. H. V. GULASEKHARAM):—From an external point T (x, y), tangents TP, TQ are drawn to the conic $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, prove that the area of the triangle TPQ is

$$\frac{S^{\frac{3}{2}}\sqrt{-\Delta}}{\Delta - CS};$$

and that the area of the quadrilateral OPTQ, where O is the centre of the conic, is

$$\tfrac{1}{2}\,\frac{\sqrt{-\Delta S}}{\Delta - CS}\left[x\,\frac{\partial S}{\partial x} + y.\frac{\partial S}{\partial y}\right],$$

where
$$\Delta = \begin{vmatrix} a, & h, & g, \\ h, & b, & f, \\ g, & f, & c, \end{vmatrix}, \text{ and } C = ab - h^2.$$

1180. (F. H. V. GULASEKHARAM):—If g and h are the lengths of the bisectors of the angles B and C respectively of a triangle ABC, prove that

$$4a^2 > g^2 + h^2.$$

1181. (C. N. SREENIVASA IYENGAR):—A′B′C′ is the triangle formed by joining the midpoints of Δ ABC; A″B″C″ is similarly formed from A′B′C′, A‴B‴C‴ from A″B″C″ and so on. If S, S′, S″, etc., N, N′, N″, etc., and O, O′, O″, etc., denote the corresponding circum-centres, N.P. centres and ortho-centres respectively, then the three sets of points all tend to the same limit. Find the exact position of this limit in relation to the triangle ABC.

1182. (B. B. BAGI):—The sides taken in order of an n-gon (n being odd) circumscribed to a circle are $a_1, a_2, a_3, \ldots a_n$. Prove that the radius of the inscribed circle is given by

$$\text{(i)}\quad \tan^{-1}\frac{x}{s - a_1 - a_3 - a_5 \ldots - a_{n-2}} + \tan^{-1}\frac{x}{s - a_2 - a_4 - a_6 - \ldots - a_{n-1}}$$

$$+ \tan^{-1} \frac{x}{s - a_3 - a_5 - a_7 - \dots - a_n} + \dots$$

$$+ \tan^{-1} \frac{x}{s - a_n - a_2 - a_4 - a_{n-3}} = \frac{\pi}{2}(n-2),$$

(ii) and that the area $\triangle$ is determined by

$$\tan^{-1} \frac{\triangle}{s(s - a_1 - a_3 - a_5 - \dots a_{n-2})}$$

$$+ \tan^{-1} \frac{\triangle}{s(s - a_2 - a_4 - \dots - a_{n-1})}$$

$$+ \tan^{-1} \frac{\triangle}{s(s - a_3 - a_5 - a_7 - \dots - a_n)} + \dots$$

$$+ \tan^{-1} \frac{\triangle}{s(s - a_n - a_2 - a_4 \dots - a_{n-3})}$$

$$= \frac{\pi}{2}(n-2),$$

where $2s = a_1 + a_2 + a_3 + \dots + a_n$.

1183. (B B. BAGI):—The circles round AQR, BRP, CPQ where P, Q and R are points in order on the sides BC, CA, AB of a triangle ABC, meet in O. If A′, B′, C′ are the middle points of the arcs QOR, ROP, POQ then show that A′B′C′ is a triangle similar to the triangle of the ex-centres of ABC, and also that A′, B′, C′ and the in-centre of ABC are concyclic.

1184. (G. S. MAHAJANI):—In any triangle, we know that

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

Similarly, in any polygon of n sides (a_1 a_2 ... a_n)

$$a_n^2 = \sum_{1}^{n-1} a_r^2 - 2 \sum a_r a_s \cos \widehat{a_r a_s},$$

r and s being unequal and taking all integral values from 1 to $(n-1)$.

Symmetrically, the sides and angles of a polygon are connected by the following relation :—

$$\sum_{1}^{n} a_r^2 - 2 \sum a_r a_s \cos \widehat{a_r a} = 0,$$

r s being unequal and taking all integral values from 1 to n.